

# SELECTIONS, PARACOMPACTNESS AND COMPACTNESS

Mitrofan M. Choban, Ekaterina P. Mihaylova, Stoyan I. Nedev

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**Abstract:** In the present paper, the Lindelöf number and the degree of compactness of spaces and of the cozero-dimensional kernel of paracompact spaces are characterized in terms of selections of lower semi-continuous closed-valued mappings into complete metrizable (or discrete) spaces.

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## Introduction

All considered spaces are assumed to be  $T_1$ -spaces. Our terminology comes, as a rule, from ([6], [7], [8], [12]).

A topological space  $X$  is called *paracompact* if  $X$  is Hausdorff and every open cover of  $X$  has a locally finite open refinement.

One of the main results of the theory of continuous selections is the following theorem:

**Theorem 0.1 (E.Michael [9])** *For any lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  of a paracompact space  $X$  into a complete metrizable space  $Y$  there exist a compact-valued lower semi-continuous mapping  $\varphi : X \rightarrow Y$  and a compact-valued upper semi-continuous mapping  $\psi : X \rightarrow Y$  such that  $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$  for any  $x \in X$ .*

*Moreover, if  $\dim X = 0$ , then the selections  $\phi, \psi$  of  $\theta$  are single-valued and continuous.*

It will be shown that the existence of upper semi-continuous selections for lower semi-continuous closed-valued mappings into a discrete spaces implies the paracompactness of the domain (see [1-5, 11]).

The aim of the present article is to determine the conditions on a space  $X$  under which for any lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  of the space  $X$  into a complete metrizable (or discrete) space  $Y$  there exists a selection  $\varphi : X \rightarrow Y$  for which the image  $\varphi(X)$  is "small" in a given sense.

A family  $\gamma$  of subsets of a space  $X$  is *star-finite* (*star-countable*) if for every element  $\Gamma \in \gamma$  the set  $\{L \in \gamma : L \cap \Gamma \neq \emptyset\}$  is finite (countable).

A topological space  $X$  is called *strongly paracompact* or *hypocompact* if  $X$  is Hausdorff and every open cover of  $X$  has a star-finite open refinement.

The cardinal number  $l(X) = \min\{m : \text{every open cover of } X \text{ has an open refinement of cardinality } \leq m\}$  is the *Lindelöf number* of  $X$ .

The cardinal number  $k(X) = \min\{m : \text{every open cover of } X \text{ has an open refinement of cardinality } < m\}$  is the *degree of the compactness* of  $X$ .

Denote by  $\tau^+$  the least cardinal number larger than the cardinal number  $\tau$ . It is obvious that  $l(X) \leq k(X) \leq l(X)^+$ .

For a space  $X$  put  $\omega(X) = \bigcup\{U : U \text{ is open in } X \text{ and } \dim U = 0\}$  and let  $c\omega(X) = X \setminus \omega(X)$  be the *cozero-dimensional kernel* of  $X$  (See [4]).

**Lemma 0.2.** *Let  $X$  be a paracompact space,  $U$  be an open subset of  $X$  and  $U \cap c\omega(X) \neq \emptyset$ . Then  $\dim cl_X(U \cap c\omega(X)) \neq 0$ .*

**Proof:** See [4].  $\square$

A family  $\xi$  of subsets of  $X$  is called  $\tau$ -centred if  $\bigcap \eta \neq \emptyset$  provided  $\eta \subseteq \xi$  and  $|\eta| < \tau$ .

**Lemma 0.3.** *Let  $X$  be a paracompact space and  $\tau$  be an infinite cardinal. Then:*

1.  $l(X) \leq \tau$  if and only if any discrete closed subset of  $X$  has cardinality  $\leq \tau$ .

2. The following assertions are equivalent:

a)  $k(X) \leq \tau$ .

b) Any discrete closed subset of  $X$  has cardinality  $< \tau$ ;

c)  $\bigcap \xi \neq \emptyset$  for any  $\tau$ -centered filter of closed subsets of  $X$ .

**Proof:** It is obvious.  $\square$

Assertions 2a and 2c are equivalent and implication  $2a \rightarrow 2b$  is true for every space  $X$ .

**Lemma 0.4.** *Let  $X$  be a metrizable space and  $\tau$  be an infinite not sequential cardinal. Then:*

1.  $l(X) \leq \tau$  if and only if  $w(X) \leq \tau$ .

2.  $k(X) \leq \tau$  if and only if  $w(X) < \tau$ .

**Proof:** It is obvious.  $\square$

## 1 On the degree of compactness of spaces

A subset  $L$  of a completely regular space  $X$  is *bounded* in  $X$  if for every continuous function  $f : X \rightarrow \mathbb{R}$  the set  $f(L)$  is bounded.

A space  $X$  is called  $\mu$ -complete if it is completely regular and the closure  $cl_X L$  of every bounded subset  $L$  of  $X$  is compact.

Every paracompact space is  $\mu$ -complete. Moreover, every Dieudonné complete space is  $\mu$ -complete (see [6]).

**Definition 1.1.** *Let  $X$  be a space and  $\tau$  be an infinite cardinal. Put  $k(X, \tau) = \bigcup\{U : U \text{ is open in } X \text{ and } k(cl_X U) < \tau\}$  and  $c(X, \tau) = X \setminus k(X, \tau)$ . For every  $x \in X$  put  $k(x, X) = \min\{k(cl_X U) : U \text{ is an open in } X \text{ neighborhood of } x\}$ .*

By definition,  $k(X, \tau) = \{x \in X : k(x, X) < \tau\}$ .

**Lemma 1.2.** *Let  $X$  be a space,  $\tau$  be an infinite cardinal,  $\{U_\alpha : \alpha \in A\}$  be an open discrete family in  $X$  and  $k(X) \leq \tau$ . Then:*

1.  $|A| < \tau$ ;
2. If  $x_\alpha \in U_\alpha \cap k(X, \tau)$  for every  $\alpha \in A$ , then  $\sup\{k(x_\alpha, X) : \alpha \in A\} < \tau$ ;
3. If  $x_\alpha \in U_\alpha \cap c(X, \tau)$  for every  $\alpha \in A$ , then  $|A| < cf(\tau)$ .

**Proof:** Since  $k(X) \leq \tau$ , every discrete family in  $X$  has cardinality  $< \tau$ .

Suppose that  $x_\alpha \in U_\alpha \cap k(X, \tau)$  for every  $\alpha \in A$  and  $\sup\{k(x_\alpha, X) : \alpha \in A\} = \tau$ . In this case  $\tau$  is a non-regular limit cardinal and  $cf(\tau) \leq |A| < \tau$ . From our assumption it follows that there exists a family of cardinals  $\{\tau_\alpha : \alpha \in A\}$  such that  $\tau_\alpha < k(x_\alpha, X)$  for every  $\alpha \in A$  and  $\sup\{\tau_\alpha : \alpha \in A\} = \tau$ . For every  $\alpha \in A$  there exists an open family  $\gamma_\alpha$  of  $X$  such that  $cl_X U_\alpha \subseteq \bigcup \gamma_\alpha$  and  $|\xi| \geq \tau_\alpha$  provided  $\xi \subseteq \gamma_\alpha$  and  $cl_X U_\alpha \subseteq \bigcup \xi$ . One can assume that  $U_\beta \cap V = \emptyset$  for every  $\alpha, \beta \in A, \alpha \neq \beta$  and  $V \in \gamma_\alpha$ . Let  $\gamma = (X \setminus \bigcup\{cl_X U_\alpha : \alpha \in A\}) \cup (\bigcup\{\gamma_\alpha : \alpha \in A\})$ . Then  $\gamma$  is an open cover of  $X$  and every subcover of  $\gamma$  has a cardinality  $\geq \sup\{\tau_\alpha : \alpha \in A\} = \tau$ , which is a contradiction.

If  $x_\alpha \in U_\alpha \cap c(X, \tau)$  for every  $\alpha \in A$  and  $|A| \geq cf(\tau)$ , then there exists a family of cardinals  $\{\tau_\alpha : \alpha \in A\}$  such that  $\tau_\alpha < \tau$  for every  $\alpha \in A$  and  $\sup\{\tau_\alpha : \alpha \in A\} = \tau$ . Since  $k(x_\alpha, X) = \tau \geq \tau_\alpha$  for every  $\alpha \in A$ , one can obtain a contradiction as in the previous case.  $\square$

**Lemma 1.3.** *Let  $X$  be a completely regular space,  $\tau$  be a sequential cardinal and  $k(X) \leq \tau$ . Then the set  $c(X, \tau)$  is closed and bounded. Moreover, if  $X$  is a  $\mu$ -complete space, then:*

1.  $c(X, \tau)$  is a compact subset;
2. If  $Y \subseteq k(X, \tau)$  is a closed subset of  $X$ , then  $k(Y) < \tau$ .

**Proof:** If  $\tau = \aleph_0$ , then the space  $X$  is compact and  $k(X, \tau)$  is the subset of all isolated in  $X$  points. Thus the set  $c(X, \tau)$  is compact and every closed in  $X$  subset of  $k(X, \tau)$  is finite.

Suppose that  $\tau$  is uncountable. There exists a family of infinite cardinal numbers  $\{\tau_n : n \in \mathbb{N}\}$  such that  $\tau_n < \tau_{n+1} < \tau$  for every  $n \in \mathbb{N}$  and  $\sup\{\tau_n : n \in \mathbb{N}\} = \tau$ . Suppose that the set  $c(X, \tau)$  is unbounded in  $X$ . Then there exists a continuous function  $f : X \rightarrow \mathbb{R}$  and a sequence  $\{x_n \in c(X, \tau) : n \in \mathbb{N}\}$  such that  $f(x_1) = 1$  and  $f(x_{n+1}) \geq 3 + f(x_n)$  for every  $n \in \mathbb{N}$ . The family  $\xi = \{U_n = f^{-1}((f(x_n) - 1, f(x_n) + 1)) : n \in \mathbb{N}\}$  is discrete in  $X$  and  $x_n \in U_n$  for every  $n \in \mathbb{N}$ . Then, by virtue of Lemma 1.2,  $|\xi| < cf(\tau) = \aleph_0$ , which is a contradiction. Thus the set  $c(X, \tau)$  is closed and bounded in  $X$ .

Assume now that  $X$  is a  $\mu$ -complete space. In this case the set  $c(X, \tau)$  is compact.

Suppose that  $Y \subseteq k(X, \tau)$  is a closed subset of  $X$  and  $k(Y) = \tau$ . We affirm that  $\sup\{k(y, X) : y \in Y\} < \tau$ . For every  $x \in k(X, \tau)$  fix a neighborhood  $U_x$  in  $X$  such that  $k(cl_X U_x) = k(x, X)$ . Suppose that  $\sup\{k(y, X) : y \in Y\} = \tau$ . For every  $n \in \mathbb{N}$  fix a point  $y_n \in Y$  such that  $k(y_n, X) \geq \tau_n$ . Put  $L = \{y_n : n \in \mathbb{N}\}$ . If the set  $L$  is unbounded in  $X$ , then there exists a continuous function  $f : X \rightarrow \mathbb{R}$  such that  $\sup\{f(y_n) : n \in \mathbb{N}\} = \infty$ . One can assume that  $f(y_{n+1}) > 3 + f(y_n)$ . The family  $\xi = \{U_n = f^{-1}((f(y_n) - 1, f(y_n) + 1)) : n \in \mathbb{N}\}$  is discrete in  $X$  and  $y_n \in U_n \cap k(X, \tau)$  for every  $n \in \mathbb{N}$ . Then, by virtue of Lemma 1.2,  $\sup\{k(y_n, X) : n \in \mathbb{N}\} < \tau$ , which is a contradiction. Thus the set

$L$  is bounded in  $X$ . Hence  $cl_X L$  is a compact subset of  $Y$  and there exists an accumulation point  $y \in cl_X L \setminus (L \setminus \{y\})$ . In this case  $y \in k(X, \tau)$ ,  $k(y, X) < \tau$  and  $k(y, X) = \sup\{k(y_n, X) : n \in \mathbb{N}\} = \tau$  which is a contradiction. Hence  $\sup\{k(y, X) : y \in Y\} \leq \tau' < \tau$ .

Since  $k(X) \leq \tau$ , then there exists a subset  $Y' \subseteq Y$  such that  $|Y'| \leq \tau''$ ,  $\tau' \leq \tau'' < \tau$  and  $Y \subseteq \bigcup\{U_y : y \in Y'\}$ . Since  $k(cl_X U_y) \leq \tau' \leq \tau''$  for every  $y \in Y'$  and  $|Y'| \leq \tau''$ , then  $l(\bigcup\{cl_X U_y : y \in Y'\}) \leq \tau'' < \tau$ . Thus  $l(Y) \leq \tau'' < \tau$  and  $k(Y) < \tau$ .  $\square$

A subspace  $Z$  of a space  $X$  is *paracompact in  $X$*  if for every open family  $\gamma = \{W_\mu : \mu \in M\}$  of  $X$ , for which  $Z \subseteq \bigcup \gamma$ , there exists an open locally finite family  $\eta = \{W'_\mu : \mu \in M\}$  of  $X$  such that  $Z \subseteq \bigcup \eta$  and  $W'_\mu \subseteq W_\mu$  for any  $\mu \in M$ .

**Lemma 1.4.** *Let  $X$  be a regular space,  $Z$  be a paracompact in  $X$  subspace,  $\tau$  be a limit cardinal number,  $k(Z) \leq \tau$  and  $k(Y) < \tau$  for every closed subspace  $Y \subseteq X \setminus Z$  of the space  $X$ . Then:*

1.  $k(X) \leq \tau$ ,  $c(X, \tau) \subseteq Z$  and  $k(c(X, \tau)) \leq cf(\tau)$ ;
2. If  $Y \subseteq k(X, \tau)$  is a closed subset of  $X$ , then  $k(Y) < \tau$ .
3.  $Z$  is a closed subspace of  $X$ .

**Proof:** Assertion 3 is obvious.

Let  $x \in X \setminus Z$ . Fix an open subset  $U$  of  $X$  such that  $x \in U \subseteq cl_X U \subseteq X \setminus Z$ . Then  $k(cl_X U) < \tau$  and  $c(X, \tau) \subseteq Z$ .

Let  $\gamma$  be an open cover of  $X$ . Since  $k(Z) \leq \tau$ , there exists a subsystem  $\xi$  of  $\gamma$  such that  $|\xi| < \tau$  and  $Z \subseteq \bigcup \xi$ . Let  $Y = X \setminus \bigcup \xi$ . Since  $k(Y) < \tau$ , there exists a subsystem  $\zeta$  of  $\gamma$  such that  $|\zeta| < \tau$  and  $Y \subseteq \bigcup \zeta$ . Put  $\eta = \zeta \cup \xi$ . Then  $\eta$  is a subcover of  $\gamma$  and  $|\eta| < \tau$ . Thus  $k(X) \leq \tau$ .

Suppose that  $k(c(X, \tau)) > cf(\tau)$ . Since  $Z$  is paracompact in  $X$ , the subspace  $c(X, \tau)$  is paracompact in  $X$  and there exists an open locally-finite family  $\{V_\alpha : \alpha \in A\}$  of  $X$  such that  $c(X, \tau) \subseteq \bigcup\{V_\alpha : \alpha \in A\}$ ,  $|A| \geq cf(\tau)$  and  $c(X, \tau) \setminus \bigcup\{V_\beta : \beta \in A \setminus \{\alpha\}\} \neq \emptyset$  for every  $\alpha \in A$ . For every  $\alpha \in A$  fix  $y_\alpha \in c(X, \tau) \setminus \bigcup\{V_\beta : \beta \in A \setminus \{\alpha\}\} \neq \emptyset$ . Then  $\{y_\alpha : \alpha \in A\}$  is a closed discrete subset of  $X$ . There exists an open discrete family  $\{W_\alpha : \alpha \in A\}$  such that  $y_\alpha \in W_\alpha \subseteq V_\alpha$  for every  $\alpha \in A$ . By virtue of Lemma 1.2,  $|A| < cf(\tau)$ , which is a contradiction. Thus  $k(c(X, \tau)) \leq cf(\tau)$ .

Fix now a closed subset  $Y$  of the space  $X$  such that  $Y \subseteq k(X, \tau)$ . We put  $S = Y \cap Z$  and  $\tau' = \sup\{k(y, X) : y \in S\}$ .

Suppose that  $\tau' = \tau$ . There exists a family of cardinals  $\{\tau_\alpha : \alpha \in A\}$  such that  $|A| = cf(\tau)$ ,  $\sup\{\tau_\alpha : \alpha \in A\} = \tau$  and  $\tau_\alpha < \tau$  for every  $\alpha \in A$ . One can assume that  $A$  is well ordered and  $\tau_\alpha < \tau_\beta$  for every  $\alpha, \beta \in A$  and  $\alpha < \beta$ . For every  $\alpha \in A$  there exists  $y_\alpha \in S$  such that  $k(y_\alpha, X) > \tau_\alpha$ . Let  $L = \{y_\alpha : \alpha \in A\}$ . The cardinal  $cf(\tau)$  is regular. If  $y \in X$  and  $|W \cap L| = |A|$  for every neighborhood  $W$  of  $y$  in  $X$ , then  $y \in Y \subseteq k(X, \tau)$ ,  $k(y, X) < \tau$  and  $k(y, X) \geq \sup\{k(y_\alpha, X) : \alpha \in A\} = \tau$ , which is a contradiction. Thus for every  $y \in X$  there exists an open neighborhood  $W_y$  of  $y$  in  $X$  such that  $|cl_X W_y \cap L| < |A| = cf(\tau)$ . There exists an open locally-finite family  $\{H_z : z \in Z\}$  of  $X$  such that  $Z \subseteq \bigcup\{H_z : z \in Z\}$  and  $H_z \subseteq W_z$  for every  $z \in Z$ . Let  $Z' = \{z \in Z : H_z \cap L \neq \emptyset\}$ . The set  $Z'$  is discrete and closed in  $X$ . Since  $Z$  is a paracompact space, we have  $|Z'| = \tau'' < \tau$  and

$|H_z \cap L| = \tau(z) < cf(\tau)$  for any  $z \in Z'$ . Thus  $|L| = |\cup\{H_z \cap L : z \in Z'\}| < \tau$ , a contradiction. Therefore  $\tau' < \tau$ .

Since  $S$  is paracompact in  $X$  and  $S \subseteq k(X, \tau)$ , there exist a set  $M \subseteq S$  and a locally finite open in  $X$  family  $\{U_\mu : \mu \in M\}$  such that  $k(cl_X U_\mu) \leq \tau'$ ,  $|M| = \tau_1 < \tau$  and  $S \subseteq \cup\{U_\mu : \mu \in M\}$ . Then  $k(S_1) = \tau_2 < \tau$ , where  $S_1 = \cup\{cl_X U_\mu : \mu \in M\}$ . Let  $Y_1 = Y \setminus \cup\{U_\mu : \mu \in M\}$ . Since  $Y_1$  is a closed subset of  $X$  and  $Y_1 \subseteq X \setminus Z$ ,  $k(Y_1) = \tau_3 < \tau$ . Thus  $k(Y) \leq k(Y_1) + k(S_1) < \tau$ .  $\square$

**Corollary 1.5.** *Let  $X$  be a paracompact space,  $\tau$  be a limit cardinal and  $k(X) \leq \tau$ . Then:*

1.  $k(c(X, \tau)) \leq cf(\tau)$ ;
2. *If  $Y \subseteq k(X, \tau)$  is a closed subset of  $X$ , then  $k(Y) < \tau$ .*

A *shrinking* of a cover  $\xi = \{U_\alpha : \alpha \in A\}$  of the space  $X$  is a cover  $\gamma = \{V_\alpha : \alpha \in A\}$  such that  $V_\alpha \subseteq U_\alpha$  for every  $\alpha \in A$  (see [6], [7]). The operation of shrinking preserves the properties of local finiteness, star-finiteness and star-countableness.

Let  $\tau$  be an infinite cardinal number. A family  $\gamma$  of subsets of a space  $X$  is called  $\tau$ -*star* ( $\tau^-$ -*star*) if  $|\{H \in \gamma : H \cap L \neq \emptyset\}| \leq \tau$  ( $|\{H \in \gamma : H \cap L \neq \emptyset\}| < \tau$ ) for every  $L \in \gamma$ .

A family  $\{H_\alpha : \alpha \in A\}$  of subsets of a space  $X$  is *closure-preserving* if  $\cup\{cl_X H_\beta : \beta \in B\} = cl_X(\cup\{H_\beta : \beta \in B\})$  for every  $B \subseteq A$  (see [10]).

**Proposition 1.6.** *Let  $\tau$  be an infinite cardinal and  $X$  be a paracompact space. Then the following assertions are equivalent:*

1.  $k(cw(X)) \leq \tau$ .
2. *For every open cover of  $X$  there exists an open  $\tau^-$ -star shrinking.*
3. *For every open cover of  $X$  there exists a closed closure-preserving  $\tau^-$ -star shrinking.*
4. *For every open cover of  $X$  there exists a closed  $\tau^-$ -star shrinking.*

**Proof:**  $(1 \Rightarrow 2)$  and  $(1 \Rightarrow 3)$  Let  $\xi = \{U_\alpha : \alpha \in A\}$  be an open cover of  $X$ . There exist a subset  $B$  of  $A$  and an open-and-closed subset  $H$  of  $X$  such that  $cw(X) \subseteq H \subseteq \cup\{U_\alpha : \alpha \in B\}$  and  $|B| < \tau$  (see the proof of Proposition 4 [4]). Since  $dim(X \setminus H) = 0$  (unless  $X \setminus H$  is empty) there exists a discrete family  $\{W_\alpha : \alpha \in A\}$  of open-and-closed subsets of  $X$  such that  $\cup\{W_\alpha : \alpha \in A\} = X \setminus H$  and  $W_\alpha \subseteq U_\alpha$  for every  $\alpha \in A$ . Let  $V_\alpha = (U_\alpha \cap H) \cup W_\alpha$  for  $\alpha \in B$  and  $V_\alpha = W_\alpha$  for  $\alpha \in A \setminus B$ . Obviously  $\gamma = \{V_\alpha : \alpha \in A\}$  is an open  $\tau^-$ -star shrinking of  $\xi$ .

Since  $X$  is paracompact, there exists a closed locally finite family  $\{H_\alpha : \alpha \in B\}$  such that  $H = \cup\{H_\alpha : \alpha \in B\}$  and  $H_\alpha \subseteq U_\alpha$  for any  $\alpha \in B$ . Put  $H_\alpha = W_\alpha$  for any  $\alpha \in A \setminus B$ . Obviously  $\lambda = \{H_\alpha : \alpha \in A\}$  is a closed locally finite  $\tau^-$ -star shrinking of  $\xi$ . Every locally finite family is closure-preserving. Implications  $(1 \Rightarrow 2)$  and  $(1 \Rightarrow 3)$  are proved.

Implication  $(3 \Rightarrow 4)$  is obvious.

$(2 \Rightarrow 1)$  and  $(4 \Rightarrow 1)$  Suppose  $k(cw(X)) > \tau$ . There exists a locally finite open cover  $\xi = \{U_\alpha : \alpha \in A\}$  of  $cw(X)$  such that  $cw(X) \setminus \cup\{U_\alpha : \alpha \in B\} \neq \emptyset$  provided  $B \subseteq A$  and  $|B| < \tau$ . One can assume that  $cw(X) \setminus \cup\{U_\alpha : \alpha \in B\} \neq \emptyset$  for every proper subset  $B$  of  $A$ . Fix a point  $x_\alpha \in cw(X) \setminus \cup\{U_\beta : \beta \in A \setminus \{\alpha\}\}$

for every  $\alpha \in A$ . The set  $\{x_\alpha : \alpha \in A\}$  is discrete in  $X$ . There exists a discrete family  $\{V_\alpha : \alpha \in A\}$  of open subsets of  $X$  such that  $x_\alpha \in V_\alpha \subseteq cl_X V_\alpha \subseteq U_\alpha$  for every  $\alpha \in A$ . Let  $X_\alpha = cl_X V_\alpha$ . Then  $dim X_\alpha > 0$  and there exist two closed disjoint subsets  $F_\alpha$  and  $P_\alpha$  of  $X_\alpha$  such that if  $W_\alpha$  and  $O_\alpha$  are open in  $X$  and  $F_\alpha \subseteq W_\alpha \subseteq X \setminus P_\alpha$ ,  $P_\alpha \subseteq O_\alpha \subseteq X \setminus F_\alpha$  and  $X_\alpha \subseteq W_\alpha \cup O_\alpha$ , then  $X_\alpha \cap W_\alpha \cap O_\alpha \neq \emptyset$ . The family  $\{F_\alpha : \alpha \in A\}$  and the family  $\{P_\alpha : \alpha \in A\}$  are discrete in  $X$ . There exists a discrete family  $\{Q_\alpha : \alpha \in A\}$  of open subsets of  $X$  such that  $(\bigcup\{Q_\alpha : \alpha \in A\}) \cap (\bigcup\{F_\alpha : \alpha \in A\}) = \emptyset$ ,  $P_\alpha \subseteq Q_\alpha$  and  $Q_\alpha \cap (\bigcup\{X_\beta : \beta \in A \setminus \{\alpha\}\}) = \emptyset$  for every  $\alpha \in A$ . Let  $\mu \notin A$ ,  $M = A \cup \{\mu\}$  and  $Q_\mu = X \setminus \bigcup\{P_\alpha : \alpha \in A\}$ . Then  $\zeta = \{Q_m : m \in M\}$  is an open cover of  $X$ . If  $\gamma = \{H_m : m \in M\}$  is an open shrinking of  $\zeta$ , then  $H_\mu \cap H_\alpha \neq \emptyset$  for every  $\alpha \in A$ . The last contradicts 2. Suppose now that  $\gamma = \{H_m : m \in M\}$  is a closed shrinking of  $\zeta$ . Let  $\alpha \in A$  and  $H_\alpha \cap H_\mu = \emptyset$ . There exist two disjoint open subsets  $W_\alpha$  and  $O_\alpha$  of  $X$  such that  $H_\alpha \subseteq W_\alpha$  and  $H_\mu \subseteq O_\alpha$ . Then  $X_\alpha \subseteq H_\alpha \cup H_\mu \subseteq O_\alpha \cup W_\alpha$ ,  $P_\alpha \subseteq W_\alpha \subseteq X \setminus F_\alpha$ ,  $F_\alpha \subseteq O_\alpha \subseteq X \setminus P_\alpha$ ,  $X_\alpha \subseteq W_\alpha \cup O_\alpha$  and  $X_\alpha \cap W_\alpha \cap O_\alpha = \emptyset$ . The last contradicts 4. Implications  $(2 \Rightarrow 1)$  and  $(4 \Rightarrow 1)$  are proved.  $\square$

## 2 The degree of compactness and selections

Let  $X$  and  $Y$  be non-empty topological spaces. A *set-valued mapping*  $\theta : X \rightarrow Y$  assigns to every  $x \in X$  a non-empty subset  $\theta(x)$  of  $Y$ . If  $\phi, \psi : X \rightarrow Y$  are set-valued mappings and  $\phi(x) \subseteq \psi(x)$  for every  $x \in X$ , then  $\phi$  is called a *selection* of  $\psi$ .

Let  $\theta : X \rightarrow Y$  be a set-valued mapping and let  $A \subseteq X$  and  $B \subseteq Y$ . The set  $\theta^{-1}(B) = \{x \in X : \theta(x) \cap B \neq \emptyset\}$  is the *inverse image* of the set  $B$ ,  $\theta(A) = \bigcup\{\theta(x) : x \in A\}$  is the *image* of the set  $A$  and  $\theta^{n+1}(A) = \theta(\theta^{-1}(\theta^n(A)))$  is the  *$n+1$ -image* of the set  $A$ . The set  $\theta^\infty(A) = \bigcup\{\theta^n(A) : n \in \mathbb{N}\}$  is the *largest image* of the set  $A$ .

A set-valued mapping  $\theta : X \rightarrow Y$  is called *lower (upper) semi-continuous* if for every open (closed) subset  $H$  of  $Y$  the set  $\theta^{-1}(H)$  is open (closed) in  $X$ .

In the present section we study the mutual relations between the following properties of topological spaces:

K1.  $k(X) \leq \tau$ .

K2. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $k(cl_Y \phi(X)) \leq \tau$ .

K3. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a set-valued selection  $g : X \rightarrow Y$  of  $\theta$  such that  $k(cl_Y g(X)) \leq \tau$ .

K4. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  of  $\theta$  such that  $k(cl_Y g(X)) \leq \tau$ .

K5. For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that

$|\phi(X)| < \tau$ .

K6. For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a set-valued selection  $g : X \rightarrow Y$  of  $\theta$  such that  $|g(X)| < \tau$ .

K7. For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  of  $\theta$  such that  $|g(X)| < \tau$ .

K8. Every open cover of  $X$  has a subcover of cardinality  $< \tau$ .

K9. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exist a compact-valued lower semi-continuous mapping  $\varphi : X \rightarrow Y$  and a compact-valued upper semi-continuous mapping  $\psi : X \rightarrow Y$  such that  $k(\text{cl}_Y(\psi(X))) \leq \tau$  and  $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$  for any  $x \in X$ .

K10. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists an upper semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $k(\text{cl}_Y \phi(X)) \leq \tau$ .

K11. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $w(\phi(X)) < \tau$ .

K12. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exist a closed-valued lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$ , a selection  $\mu : X \rightarrow Y$  of  $\theta$  and a closed  $G_\delta$  subset  $F$  of the space  $X$  such that:

- $\phi(x) \subseteq \mu(x)$  for any  $x \in X$ ;
- $\phi(x) = \mu(x)$  for any  $x \in X \setminus F$ ;
- the mapping  $\mu|_F : F \rightarrow Y$  is upper semi-continuous and is closed-valued;
- $c(X, \tau) \subseteq F$ ,  $\Phi = \mu(F)$  is a compact subset of  $Y$  and  $k(Z \cap \mu(X)) < \tau$  provided  $Z \subseteq Y \setminus \Phi$  and  $Z$  is a closed subspace of the space  $Y$ ;
- $k(\text{cl}_Y \phi(X)) \leq k(\mu(X)) \leq \tau$ .

Let us mention that, in the conditions of K12:

- $k(\mu(X)) \leq \tau$  provided the set  $\Phi$  is compact and  $k(Z \cap \mu(X)) < \tau$  for a closed subset  $Z \subseteq Y \setminus \Phi$  of the space  $Y$ ;
- the mapping  $\mu : X \rightarrow Y$  is closed-valued and the mapping  $\mu|_F : F \rightarrow Y$  is compact-valued;
- the mapping  $\mu|(X \setminus F) : X \setminus F \rightarrow Y$  is lower semi-continuous;
- the mapping  $\mu : X \rightarrow Y$  is Borel measurable, i.e.  $\mu^{-1}(H)$  is a Borel subset of the space  $X$  for any open or closed subset  $H$  of  $Y$ .

The  $\sigma$ -algebra generated by the open subsets of the space  $X$  is the algebra of Borel subsets of the space  $X$ .

**Lemma 2.1.** *Let  $X$  be a space and  $\tau$  be an infinite cardinal. Then the following implications  $(K9 \rightarrow K2 \rightarrow K3 \rightarrow K4 \rightarrow K3 \rightarrow K6 \rightarrow K7 \rightarrow K8 \rightarrow K1 \rightarrow K5 \rightarrow K6, K10 \rightarrow K3)$  and  $(K12 \rightarrow K11 \rightarrow K2 \rightarrow K5 \rightarrow K6)$  are true.*

**Proof:** Implications  $(K12 \rightarrow K11 \rightarrow K2 \rightarrow K5 \rightarrow K6, K9 \rightarrow K2 \rightarrow K3 \rightarrow K6)$ ,  $(K4 \rightarrow K7, K4 \rightarrow K3)$ ,  $(K7 \rightarrow K6, K8 \rightarrow K1 \rightarrow K8)$  and  $(K10 \rightarrow K3)$  are obvious.

Let  $\phi : X \rightarrow Y$  be a set-valued selection of the mapping  $\theta : X \rightarrow Y$  and  $k(cl_Y \phi(X)) \leq \tau$ . For every  $x \in X$  fix a point  $f(x) \in \phi(x)$ . Then  $f : X \rightarrow Y$  is a single-valued selection of  $\theta$  and  $\phi, f(X) \subseteq \phi(X)$  and  $k(cl_Y f(X)) \leq k(cl_Y \phi(X)) \leq \tau$ . The implications  $(K3 \rightarrow K4)$  and  $(K6 \rightarrow K7)$  are proved.

Let  $\gamma = \{U_\alpha : \alpha \in A\}$  be an open cover of  $X$ . One may assume that  $A$  is a discrete space. For every  $x \in X$  put  $\theta_\gamma(x) = \{\alpha \in A : x \in U_\alpha\}$ . Since  $\theta^{-1}(\{\alpha\}) = U_\alpha$ , the mapping  $\theta_\gamma$  is lower semi-continuous. Let  $\phi : X \rightarrow Y$  be a set-valued selection of  $\theta_\gamma$  and  $|\phi(X)| < \tau$ . Put  $B = \phi(X)$  and  $H_\alpha = \phi^{-1}(\alpha)$  for every  $\alpha \in B$ . Then  $H_\alpha \subseteq \theta^{-1}(\{\alpha\}) = U_\alpha$  for every  $\alpha \in B$ ,  $X = \bigcup \{H_\alpha : \alpha \in B\}$ ,  $\xi = \{H_\alpha : \alpha \in B\}$  is a refinement of  $\gamma$  and  $|B| < \tau$ . Implications  $(K3 \rightarrow K8)$  and  $(K6 \rightarrow K8)$  are proved.

Let  $k(X) \leq \tau$  and  $\theta : X \rightarrow Y$  be a lower semi-continuous mapping into a discrete space  $Y$ . Then  $\{U_y = \theta^{-1}(y) : y \in Y\}$  is an open cover of  $X$ . There exists a subset  $Z \subseteq Y$  such that  $|Z| < \tau$  and  $X = \bigcup \{U_y : y \in Z\}$ . Now we put  $\phi(x) = \{y \in Z : x \in U_y\}$ . Then  $\phi : X \rightarrow Y$  is a lower semi-continuous selection of  $\theta$ ,  $\phi(x) = Z \cap \theta(x)$  for every  $x \in X$  and  $|\phi(X)| = |Z| < \tau$ . Implication  $(K1 \rightarrow K5)$  is proved. The proof is complete.

**Proposition 2.2.** *Let  $X$  be a space,  $\tau$  be an infinite cardinal and  $\theta : X \rightarrow Y$  be an upper semi-continuous mapping onto  $Y$ . Then:*

1. *If  $l(X) \leq \tau$  and  $l(\theta(x)) \leq \tau$  for every  $x \in X$ , then  $l(Y) \leq \tau$ ;*
2. *If  $k(X) \leq \tau$  and  $k(\theta(x)) \leq cf(\tau)$  for every  $x \in X$ , then  $k(Y) \leq \tau$ ;*
3. *If  $\theta$  is compact-valued, then  $l(Y) \leq l(X)$  and  $k(Y) \leq k(X)$ .*
4. *If  $X$  is a  $\mu$ -complete space,  $\tau$  is a sequential cardinal number and  $\theta$  is compact-valued, then  $c(Y, \tau) \subseteq \theta(c(X, \tau))$  and  $k(Z) < \tau$  provided  $Z \subseteq Y \setminus c(Y, \tau)$  and  $Z$  is closed in the space  $Y$ .*

**Proof:** If  $V$  is an open subset of  $Y$ , then  $\theta^*(V) = \{x \in X : \theta(x) \subseteq V\}$  is open in  $X$ .

1. Let  $\tau$  be an infinite cardinal,  $l(X) \leq \tau$  and  $l(\theta(x)) \leq \tau$  for every  $x \in X$ . Let  $\gamma = \{V_\alpha : \alpha \in A\}$  be an open cover of  $Y$ . If  $x \in X$ , then  $l(\theta(x)) \leq \tau$ . Thus every open family in  $Y$ , which covers  $\theta(x)$ , has a subfamily of cardinality  $\leq \tau$  covering  $\theta(x)$ . Hence there exists a subset  $A_x \subseteq A$  such that  $|A_x| = \tau_x \leq \tau$  and  $\theta(x) \subseteq \bigcup \{V_\alpha : \alpha \in A_x\}$ . We put  $W_x = \bigcup \{V_\alpha : \alpha \in A_x\}$  and  $U_x = \{z \in X : \theta(z) \subseteq W_x\}$ .

Obviously  $\lambda = \{U_x : x \in X\}$  is an open cover of  $X$ . Since  $l(X) \leq \tau$ , there exists an open subcover  $\zeta = \{U_x : x \in X'\}$  of  $\lambda$  such that  $|X'| \leq \tau$  and  $X' \subseteq X$ . Let  $B = \bigcup \{A_x : x \in X'\}$ . Obviously  $|B| \leq \tau$ . Since  $\theta(U_x) \subseteq W_x$  for any  $x \in X$ , we have  $Y = \theta(X) = \theta(\bigcup \{U_x : x \in X'\}) = \bigcup \{\theta(U_x) : x \in X'\} \subseteq \bigcup \{W_x : x \in X'\} = \bigcup \{V_\alpha : \alpha \in B\}$ . Hence  $\gamma' = \{V_\alpha : \alpha \in B\}$  is a subcover of  $\gamma$  of cardinality  $\leq \tau$ . Assertion 1 is proved.

2. One can follow the proof of the previous assertion 1. Let  $\tau$  be an infinite cardinal,  $k(X) \leq \tau$  and  $k(\theta(x)) \leq cf(\tau)$  for every  $x \in X$ . Let  $\gamma = \{V_\alpha : \alpha \in A\}$  be an open cover of  $Y$ . For any  $x \in X$  there exists a subset  $A_x \subseteq A$  such that  $|A_x| = \tau_x < cf(\tau)$  and  $\theta(x) \subseteq \bigcup \{V_\alpha : \alpha \in A_x\}$ . We put  $W_x = \bigcup \{V_\alpha : \alpha \in A_x\}$  and  $U_x = \{z \in X : \theta(z) \subseteq W_x\}$ .

Obviously  $\lambda = \{U_x : x \in X\}$  is an open cover of  $X$ . Since  $k(X) \leq \tau$ , there exists an open subcover  $\zeta = \{U_x : x \in X'\}$  of  $\lambda$  such that  $|X'| = \tau_0 < \tau$  and



$X' \subseteq X$ . Let  $B = \cup\{A_x : x \in X'\}$ . Since  $\theta(U_x) \subseteq W_x$  for any  $x \in X$ , we have  $Y = \theta(X) = \theta(\cup\{U_x : x \in X'\}) = \cup\{\theta(U_x) : x \in X'\} \subseteq \cup\{W_x : x \in X'\} = \cup\{V_\alpha : \alpha \in B\}$ . Hence  $\gamma' = \{V_\alpha : \alpha \in B\}$  is a subcover of  $\gamma$ .

We affirm that  $|B| < \tau$ .

Consider the following cases:

Case 1.  $\tau$  is regular, i.e.  $cf(\tau) = \tau$ .

Since  $|X'| = \tau_0 < \tau = cf(\tau)$  and  $|A_x| < \tau$  for every  $x \in X$ , it follows that  $|B| \leq \Sigma\{\tau_x : x \in X'\} = \tau' < \tau$ .

Hence  $\gamma' = \{V_\alpha : \alpha \in B\}$  has cardinality  $< \tau$ .

Case 2.  $\tau$  is not regular, i.e.  $cf(\tau) = m < \tau$ .

In this case  $\tau$  is a limit cardinal,  $\tau_0 < \tau$  and  $m < \tau$ . Hence  $\tau' = \sup\{m, \tau_0\} < \tau$ .

Since  $|A_x| = \tau_x < m$  for every  $x \in X$ , it follows that  $|B| \leq \Sigma\{\tau_x : x \in X'\} \leq \tau' < \tau$ .

Hence  $\gamma' = \{V_\alpha : \alpha \in B\}$  has cardinality  $< \tau$ .

Assertion 2 is proved.

3. Assertion 3 follows easily from assertions 1 and 2.

4. Obviously,  $\Phi = \theta(c(X, \tau))$  and  $c(Y, \tau)$  are compact subsets of the space  $Y$ . Let  $Z \subseteq Y \setminus \Phi$  be a closed subspace of the space  $Y$ . Then  $X_1 = \theta^{-1}(Z)$  is a closed subspace of the space  $X$  and  $X_1 \cap c(X, \tau) = \emptyset$ . By virtue of Lemma 1.3,  $k(X_1) < \tau$ . Let  $Y_1 = \theta(X_1)$ . Then  $\theta_1 = \theta|_{X_1} : X_1 \rightarrow Y_1$  is an upper semi-continuous mapping onto  $Y_1$ . From assertion 2 it follows that  $k(Y_1) \leq k(X_1) < \tau$ . Since  $Z$  is a closed subspace of the space  $Y_1$ , we have  $k(Z) \leq k(Y_1) < \tau$ . In particular,  $Y \setminus \Phi \subseteq k(Y, \tau)$  and  $c(Y, \tau) \subseteq \Phi$ . Since  $\Phi$  is a compact subset of  $Y$ ,  $k(\Phi) < \tau$  provided  $Z \subseteq Y \setminus c(Y, \tau)$  and  $Z$  is closed in the space  $Y$ .  $\square$

**Theorem 2.3.** *Let  $X$  be a regular space and  $\tau$  be a regular cardinal number. Then assertions K1 – K8 and K12 are equivalent. Moreover, if the cardinal number  $\tau$  is regular and uncountable, then assertions K1 – K8, K11 and K12 are equivalent.*

**Proof:** Let  $k(X) \leq \tau$  and  $\theta : X \rightarrow Y$  be a lower semi-continuous closed-valued mapping into a complete metric space  $(Y, \rho)$ .

Case 1.  $\tau = \aleph_0$ .

In this case the space  $X$  is compact. Thus, from E. Michael's Theorem [9] (see Theorem 0.1), it follows that there exist a lower semi-continuous compact-valued mapping  $\varphi : X \rightarrow Y$  and an upper semi-continuous compact-valued mapping  $\psi : X \rightarrow Y$  such that  $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$  for any  $x \in X$ . The set  $\psi(X)$  is compact and  $\varphi(X) \subseteq \psi(X)$ . Implication  $(K1 \Rightarrow K9)$  is proved.

Case 2.  $\tau > \aleph_0$ .

There exists a sequence  $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$  of open covers of the space  $X$ , a sequence  $\xi = \{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$  of open families of the space  $Y$  and a sequence  $\pi = \{\pi_n : A_{n+1} \rightarrow A_n : n \in \mathbb{N}\}$  of mappings such that:

- $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha \subseteq cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$  for any  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;
- $\cup\{cl_Y V_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$  and  $diam(V_\alpha) < 2^{-n}$  for any  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;
- $|A_n| < \tau$  for any  $n \in \mathbb{N}$ .

Let  $\eta = \{V : V \text{ is open in } Y \text{ and } \text{diam}(V) < 2^{-1}\}$ . Let  $\gamma' = \{U : U \text{ is open in } X \text{ and } \text{cl}_X U \subseteq \theta^{-1}(V) \text{ for some } V \in \eta\}$ . Since  $k(X) \leq \tau$ , there exists an open subcover  $\gamma_1 = \{U_\alpha : \alpha \in A_1\}$  of  $\gamma'$  such that  $|A_1| < \tau$ . For any  $\alpha \in A_1$  fix  $V_\alpha \in \eta$  such that  $\text{cl}_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$ .

Consider that the objects  $\{\gamma_i, \xi_i, \pi_{i-1} : i \leq n\}$  are constructed. Fix  $\alpha \in A_n$ . Let  $\eta_\alpha = \{V : V \text{ is open in } Y, \text{cl}_Y V \subseteq V_\alpha \text{ and } \text{diam}(V) < 2^{-n-1}\}$ . Let  $\gamma'_\alpha = \{W : W \text{ is open in } X \text{ and } \text{cl}_X W \subseteq \theta^{-1}(V) \text{ for some } V \in \eta_\alpha\}$ . Since  $k(\text{cl}_X U_\alpha) \leq \tau$  and  $\text{cl}_X U_\alpha \subseteq \cup \gamma'_\alpha$ , there exists an open subfamily  $\gamma_\alpha = \{W_\beta : \beta \in A_\alpha\}$  of  $\gamma'_\alpha$  such that  $|A_\alpha| < \tau$  and  $\text{cl}_X U_\alpha \subseteq \cup \{W_\beta : \beta \in A_\alpha\}$ . For any  $\beta \in A_\alpha$  fix  $V_\beta \in \eta_\alpha$  such that  $\text{cl}_X W_\beta \subseteq \theta^{-1}(V_\beta)$ . Let  $A_{n+1} = \cup \{A_\alpha : \alpha \in A_n\}$ ,  $\pi_n^{-1}(\alpha) = A_\alpha$  and  $U_\beta = U_\alpha \cap W_\beta$  for all  $\alpha \in A_n$  and  $\beta \in A_\alpha$ . Since  $\tau$  is regular and uncountable, then  $|A_{n+1}| < \tau$ .

The objects  $\{\gamma_n, \xi_n, \pi_n : n \in \mathbb{N}\}$  are constructed.

Let  $x \in X$ . Denote by  $A(x)$  the set of all sequences  $\alpha = (\alpha_n : n \in \mathbb{N})$  for which  $\alpha_n \in A_n$  and  $x \in U_{\alpha_n}$  for any  $n \in \mathbb{N}$ . For any  $\alpha = (\alpha_n : n \in \mathbb{N}) \in A(x)$  there exists a unique point  $y(\alpha) \in Y$  such that  $\{y(\alpha)\} = \cap \{V_{\alpha_n} : n \in \mathbb{N}\}$ . It is obvious that  $y(\alpha) \in \theta(x)$ . Let  $\phi(x) = \{y(\alpha) : \alpha \in A(x)\}$ . Then  $\phi$  is a selection of  $\theta$ . By construction:

- $U_\alpha \subseteq \phi^{-1}(V_\alpha)$  for all  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;
- the mapping  $\phi$  is lower semi-continuous;
- if  $Z = \phi(X)$ , then  $\{H_\alpha = Z \cap V_\alpha : \alpha \in A = \cup \{A_n : n \in \mathbb{N}\}\}$  is an open base of the subspace  $Z$ .

We affirm that  $w(Z) < \tau$ .

Subcase 2.1.  $\tau$  is a limit cardinal.

In this subcase  $m = \sup\{|A_n| : n \in \mathbb{N}\} < \tau$  and  $w(Z) \leq |A| \leq m < \tau$ .

Subcase 2.2.  $\tau$  is not a limit cardinal.

In this subcase there exists a cardinal number  $m$  such that  $m^+ = \tau$  and  $|A| \leq m$ . Thus  $w(Z) < \tau$ .

In this case we have proved implication  $(K1 \rightarrow K11)$ .

Lemma 2.1 completes the proof of the theorem.  $\square$

**Corollary 2.4.** *Let  $X$  be a regular space and  $\tau$  be a cardinal number. Then the following assertions are equivalent:*

- L1.  $l(X) \leq \tau$ .
- L2. *For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $l(\text{cl}_Y \phi(X)) \leq \tau$ .*
- L3. *For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a set-valued selection  $g : X \rightarrow Y$  of  $\theta$  such that  $l(\text{cl}_Y g(X)) \leq \tau$ .*
- L4. *For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  of  $\theta$  such that  $l(\text{cl}_Y g(X)) \leq \tau$ .*
- L5. *For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $|\phi(X)| \leq \tau$ .*

L6. For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a set-valued selection  $g : X \rightarrow Y$  of  $\theta$  such that  $|g(X)| \leq \tau$ .

L7. For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  of  $\theta$  such that  $|g(X)| \leq \tau$ .

L8. Every open cover of  $X$  has a subcover of cardinality  $\leq \tau$ .

**Proof:** Let  $l(X) \leq \tau$ . Then  $k(X) \leq \tau^+$  and  $\tau^+$  is a regular cardinal. Theorem 2.3 completes the proof.  $\square$

**Theorem 2.5.** Let  $X$  be a regular space,  $F$  be a compact subset of  $X$ ,  $\tau$  be a cardinal number and  $k(Y) < \tau$  for any closed subset  $Y \subseteq X \setminus F$  of  $X$ . Then assertions K1 – K8 and K12 are equivalent. Moreover, if the cardinal number  $\tau$  is not sequential, then assertions K1 – K8, K11 and K12 are equivalent.

**Proof:** Let  $k(X) \leq \tau$ ,  $F$  be a compact subset of  $X$ ,  $\tau$  be a cardinal number and  $k(Y) < \tau$  for any closed subset  $Y \subseteq X \setminus F$  of  $X$  and  $\theta : X \rightarrow Y$  be a lower semi-continuous closed-valued mapping into a complete metric space  $(Y, \rho)$ .

Case 1.  $\tau = \aleph_0$ .

In this case the space  $X$  is compact. Thus, from Theorem 0.1, it follows that there exist a lower semi-continuous compact-valued mapping  $\varphi : X \rightarrow Y$  and an upper semi-continuous compact-valued mapping  $\psi : X \rightarrow Y$  such that  $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$  for any  $x \in X$ . The set  $\psi(X)$  is compact and  $\varphi(X) \subseteq \psi(X)$ . Implications  $(K1 \Rightarrow K9)$  and  $(K1 \Rightarrow K12)$  are proved.

Case 2.  $\tau$  is a regular cardinal number.

In this case Theorem 2.3 completes the proof.

Case 3.  $\tau$  is an uncountable limit cardinal.

Let  $\tau' = cf(\tau)$ .

The subspace  $F$  is compact. Thus, from the E. Michael's Theorem 0.1, it follows that there exists an upper semi-continuous compact-valued mapping  $\psi : F \rightarrow Y$  such that  $\psi(x) \subseteq \theta(x)$  for any  $x \in F$ . The set  $\Phi = \psi(F)$  is compact. There exists a sequence  $\{H_n : n \in \mathbb{N}\}$  of open subsets of  $Y$  such that:

- $\Phi \subseteq H_{n+1} \subseteq cl_Y H_{n+1} \subseteq H_n$  for any  $n \in \mathbb{N}$ ;
- for every open subset  $V \supseteq \Phi$  of  $Y$  there exists  $n \in \mathbb{N}$  such that  $H_n \subseteq V$ .

There exist a sequence  $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$  of open covers of the space  $X$ , a sequence  $\xi = \{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$  of open families of the space  $Y$ , a sequence  $\{U_n : n \in \mathbb{N}\}$  of open subsets of  $X$ , a sequence  $\pi = \{\pi_n : A_{n+1} \rightarrow A_n : n \in \mathbb{N}\}$  of mappings and a sequence  $\{\tau_n : n \in \mathbb{N}\}$  of cardinal numbers such that:

- $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha \subseteq cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$  for any  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;
- $\cup\{cl_Y V_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$  and  $diam(V_\alpha) < 2^{-n}$  for any  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;
- $|A_n| < \tau$  for any  $n \in \mathbb{N}$ ;
- if  $A'_n = \{\alpha \in A_n : F \cap cl_X U_\alpha = \emptyset\}$  and  $A''_n = A_n \setminus A'_n$ , then the set  $A''_n$  is finite and  $F \subseteq U_n \subseteq cl_X U_n \subseteq \cup\{U_\alpha : \alpha \in A''_n\}$ ;
- $\tau_n \leq \tau_{n+1} < \tau$  for any  $n \in \mathbb{N}$ ;
- $cl_X U_n \subseteq \theta^{-1}(H_n)$  and  $|\{\alpha \in A_n : U_\alpha \setminus U_m \neq \emptyset\}| \leq \tau_m$  for all  $n, m \in \mathbb{N}$ ;
- $cl_X U_n \cap cl_X U_\alpha = \emptyset$  for any  $n \in \mathbb{N}$  and  $\alpha \in A'_n$ .

Let  $\eta = \{V : V \text{ is open in } Y \text{ and } \text{diam}(V) < 2^{-1}\}$ . There exists a finite subfamily  $\{V_\beta : \beta \in B_1\}$  of  $\eta$  such that  $\Phi \subseteq \cup\{V_\beta : \beta \in B_1\} \subseteq H_1$ . Let  $W_1$  be an open subset of  $Y$  and  $\Phi \subseteq W_1 \subseteq \text{cl}_Y W_1 \subseteq \cup\{V_\alpha : \alpha \in B_1\}$ .

Let  $\gamma' = \{U : U \text{ is open in } X \text{ and } \text{cl}_X U \subseteq \theta^{-1}(V) \text{ for some } V \in \eta \text{ and } U \subseteq X \setminus U_1\}$  and  $\gamma'' = \{U : U \text{ is open in } X \text{ and } \text{cl}_X U \subseteq \theta^{-1}(V_\beta) \text{ for some } \beta \in B_1\}$ .

Since  $F$  is compact, there exist a finite family  $\gamma'_1 = \{U_\alpha : \alpha \in A'_1\}$  of  $\gamma'$  and an open subset  $U_1$  of  $X$  such that  $F \subseteq U_1 \subseteq \text{cl}_X U_1 \subseteq \cup\{U_\alpha : \alpha \in A'_1\}$  and  $F \cap U_\alpha \neq \emptyset$  for any  $\alpha \in A'_1$ . For every  $\alpha \in A'_1$  fix  $V_\alpha = V_\beta$  for some  $\beta \in B_1$  such that  $\text{cl} U_\alpha \subseteq \theta^{-1}(V_\alpha)$ . Let  $Y_1 = X \setminus U_1$ . Since  $k(Y) = \tau'_1 < \tau$ , there exists an open subfamily  $\gamma'_1 = \{U_\alpha : \alpha \in A'_1\}$  of  $\gamma'$  such that  $|A'_1| \leq \tau_1$ ,  $Y_1 \subseteq \cup\{U_\alpha : \alpha \in A'_1\}$  and  $\text{cl}_X U_1 \cap (\cup\{\text{cl}_X U_\alpha : \alpha \in A'_1\}) = \emptyset$ . For any  $\alpha \in A'_1$  fix  $V_\alpha \in \eta'$  such that  $\text{cl}_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$ . Let  $A_1 = A'_1 \cup A''_1$ ,  $\gamma_1 = \{U_\alpha : \alpha \in A_1\}$  and  $\eta_1 = \{V_\alpha : \alpha \in A_1\}$ .

Consider that the objects  $\{\gamma_i, \xi_i, \pi_{i-1}, U_i, \tau_i, : i \leq n\}$  are constructed.

We put  $A_{im} = \{\alpha \in A_i : U_\alpha \cap U_m \neq \emptyset\}$  for all  $i, m \leq n$ .

Fix  $\alpha \in A_n$ .

Let  $\eta_\alpha = \{V : V \text{ is open in } Y, \text{cl}_Y V \subseteq V_\alpha \text{ and } \text{diam}(V) < 2^{-n-1}\}$  and  $\gamma'_\alpha = \{W : W \text{ is open in } X \text{ and } \text{cl}_X W \subseteq \theta^{-1}(V) \text{ for some } V \in \eta_\alpha\}$ .

Assume that  $\alpha \in A''_n$ .

Since  $F_\alpha = F \cap \text{cl}_X U_\alpha$  is a compact subset of  $X$  there exists a finite subfamily  $\gamma_{0\alpha} = \{W_\beta : \beta \in A''_{0\alpha}\}$  of  $\gamma'_\alpha$  such that  $F_\alpha \subseteq \cup\{W_\beta : \beta \in A''_{0\alpha}\}$ ,  $F_\alpha \cap W_\beta \neq \emptyset$  for any  $\beta \in A''_{0\alpha}$  and for any  $\beta \in A''_{0\alpha}$  there exists  $V_\beta \in \eta_\alpha$  such that  $V_\beta \subseteq H_{n+1}$  and  $\text{cl}_X W_\beta \subseteq \theta^{-1}(V_\beta)$ . Now we put  $U_\beta = W_\beta \cap U_\alpha$ .

Let  $A''_{n+1} = \cup\{A_{0\alpha} : \beta \in A''_n\}$ ,  $\gamma''_{n+1} = \{U_\beta : \beta \in A''_{n+1}\}$  and  $\eta''_{n+1} = \{V_\beta : \beta \in A''_{n+1}\}$ .

Let  $\Phi_\alpha = \text{cl}_X U_\alpha \setminus \cup\{U_\beta : \beta \in A''_{0\alpha}\}$  and  $U'_n = U_n \setminus \cup\{\Phi_\alpha : \alpha \in A''_n\}$ . Then  $U'_n$  is an open subset of  $X$  and  $F \subseteq U'_n \cap \cup(\{U_\beta : \beta \in A''_{0\alpha}\})$ .

There exists an open subset  $U_{n+1}$  of  $X$  such that  $U_{n+1} \subseteq \text{cl}_X U_{n+1} \subseteq U'_n \cap U_n \cap (\cup\{U_\beta : \beta \in A''_{0\alpha}\})$ .

Let  $Y_i = X \setminus U_i$  for any  $i \leq n+1$ . Then  $\tau_i = k(Y_i)$  for any  $i \leq n+1$ .

For any  $\alpha \in A_n$  there exist the subfamilies  $\gamma'_{i\alpha} = \{W_\beta : \beta \in A'_{i\alpha}\}$ ,  $i \leq n+1$ , of  $\gamma'_\alpha$  and the subfamilies  $\eta'_{i\alpha} = \{V_\beta : \beta \in A'_{i\alpha}\}$ ,  $i \leq n+1$ , of  $\eta'_\alpha$  such that:

- $|A'_{i\alpha}| < \tau_i$  for any  $i \leq n+1$ ;
- $Y_i \cap \text{cl}_X U_\alpha \subseteq \cup\{W_\beta : \beta \in \cup\{A'_{jn\alpha} : j \leq i\}\}$  for any  $i \leq n+1$ ;
- $Y_i \cap (\cup\{W_\beta : \beta \in \cup\{A'_{jn\alpha} : i < j \leq n+1\}\}) = \emptyset$  for any  $i < n+1$ .

Now we put  $A_{n\alpha} = \cup\{A'_{i\alpha} : 0 \leq i \leq n+1\}$ ,  $A_{n+1} = \cup\{A_{n\alpha} : \alpha \in A_n\}$ ,  $U_\beta = W_\beta \cap U_\alpha$ ,  $\gamma_{n+1} = \{U_\beta : \beta \in A_{n+1}\}$ ,  $\eta_{n+1} = \{V_\beta : \beta \in A_{n+1}\}$  and  $\pi_{n+1}^{-1}(\alpha) = A_{n\alpha}$ .

The objects  $\{\gamma_n, \xi_n, \pi_n, U_n, \tau_n : n \in \mathbb{N}\}$  are constructed.

Let  $x \in X$ . Denote by  $A(x)$  the set of all sequences  $\alpha = (\alpha_n : n \in \mathbb{N})$  for which  $\alpha_n \in A_n$  and  $x \in U_{\alpha_n}$  for any  $n \in \mathbb{N}$ . For any  $\alpha = (\alpha_n : n \in \mathbb{N}) \in A(x)$  there exists a unique point  $y(\alpha) \in Y$  such that  $\{y(\alpha)\} = \cap\{V_{\alpha_n} : n \in \mathbb{N}\}$ . It is obvious that  $y(\alpha) \in \theta(x)$ . Let  $\phi(x) = \{y(\alpha) : \alpha \in A(x)\}$ . Then  $\phi$  is a selection of  $\theta$ . By construction:

- $U_\alpha \subseteq \phi^{-1}(V_\alpha)$  for all  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;

- the mapping  $\phi$  is lower semi-continuous;
- if  $Z = \phi(X)$ , then  $\{H_\alpha = Z \cap V_\alpha : \alpha \in A = \cup\{A_n : n \in \mathbb{N}\}\}$  is an open base of the subspace  $Z$ .

We affirm that  $k(cl_Y Z) \leq \tau$ .

Subcase 3.1.  $\tau$  is not a sequential cardinal.

In this subcase  $m = \sup\{|A_n| : n \in \mathbb{N}\} < \tau$  and  $w(Z) \leq |A| \leq m < \tau$ . In this subcase we are proved the implication  $(K1 \rightarrow K11)$ .

Subcase 3.2.  $\tau$  is a sequential cardinal.

Let  $Z_n = \phi(Y_n)$  and  $A_{nk} = \{\alpha \in A_n : Y_k \cap U_\alpha \neq \emptyset\}$ . Then  $|A_{nk}| < \tau_n$  for all  $n, k \in \mathbb{N}$ . Thus  $w(Z_n) < \tau_n$ .

Since  $\phi(X) \setminus Z_n \subseteq H_n$ , we have  $k(cl_Y \phi(X)) \leq \tau$ .

In this subcase we have proved implication  $(K1 \rightarrow K2)$ .

Let  $H = \cap\{U_n : n \in \mathbb{N}\}$ ,  $\mu = \{\mu_n = \{H \cap U_\alpha : \alpha \in A_n''\} \text{ and } q\{q_n = \pi_n|A_{n+1}'' : A_{n+1}'' \rightarrow A_n'' : n \in \mathbb{N}\}$ . By construction, we have  $\cup\{W_\beta; \beta \in q_n^{-1}(\alpha)\} = W_\alpha \subseteq cl_X W_\alpha \subseteq \theta^{-1}(V_\alpha)$  for any  $\alpha \in A_n''$  and  $n \in \mathbb{N}$ . Let  $x \in H$ . Denote by  $B(x)$  the set of all sequences  $\alpha = (\alpha_n : n \in \mathbb{N})$  for which  $\alpha_n \in A_n''$  and  $x \in cl_X W_{\alpha_n}$  for any  $n \in \mathbb{N}$ . For any  $\alpha = (\alpha_n : n \in \mathbb{N}) \in B(x)$  there exists a unique point  $y(\alpha) \in Y$  such that  $\{y(\alpha)\} = \cap\{V_{\alpha_n} : n \in \mathbb{N}\}$ . It is obvious that  $y(\alpha) \in \Phi \cap \theta(x)$ . Let  $\mu_1(x) = \{y(\alpha) : \alpha \in B(x)\}$ . The mapping  $\mu_1 : H \rightarrow \Phi$  is compact-valued and upper semi-continuous. Let  $\mu(x) = \phi(x)$  for  $x \in X \setminus H$  and  $\mu(x) = \mu_1(x)$  for  $x \in H$ . Then  $\mu$  is a selection of  $\theta$ . Fix a closed subset  $Z \subseteq Y \setminus \Phi$  of the space  $Y$ . Then  $Z \cap \mu(X) \subseteq \phi(Y_n)$  for some  $n \in \mathbb{N}$ . Thus  $w(Z \cap \mu(X)) < \tau$ . In this subcase we have proved implication  $(K1 \rightarrow K12)$ , too.

Lemma 2.1 completes the proof of the theorem.  $\square$

The last theorem and Lemma 1.3 imply

**Corollary 2.6.** *Let  $X$  be a  $\mu$ -complete space and  $\tau$  be a sequential cardinal number. Then assertions  $K1 - K8$  are equivalent.*

Theorem 2.5 is signigative for a sequential cardinal  $\tau$ . Every compact subset of  $X$  is paracompact in  $X$ . In fact we have

**Theorem 2.7.** *Let  $X$  be a regular space,  $F$  be a paracompact in  $X$  subspace,  $\tau$  be an infinite cardinal number,  $k(F) \leq \tau$ ,  $k(Y) < \tau$  for any closed subset  $Y \subseteq X \setminus F$  of  $X$ . Then assertions  $K1 - K8$  are equivalent. Moreover, if the cardinal number  $\tau$  is not sequential, then assertions  $K1 - K8$  and  $K11$  are equivalent.*

**Proof:** It is obvious that for any open in  $X$  set  $U \supseteq F$  there exists an open subset  $V$  of  $X$  such that  $F \subseteq U \subseteq cl_X U \subseteq V$

Case 1.  $\tau$  is a regular cardinal number.

In this case Theorem 2.3 completes the proof.

Case 2.  $\tau$  is a sequential cardinal number.

In this case Theorem 2.5 and Lemma 1.4 complete the proof.

Case 3.  $\tau$  be a limit non-sequential cardinal.

Let  $\tau^* = cf(\tau) < \tau$ . Obviously,  $\tau^*$  is a regular cardinal and  $\tau^* < \tau$ .

There exist a sequence  $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$  of open covers of the space  $X$ , a sequence  $\xi = \{\xi_n = \{V_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$  of open families of the space  $Y$ , a sequence  $\{U_n : n \in \mathbb{N}\}$  of open subsets of  $X$  a sequence

$\pi = \{\pi_n : A_{n+1} \rightarrow A_n : n \in \mathbb{N}\}$  of mappings and a sequence  $\{\tau_n : n \in \mathbb{N}\}$  of cardinal numbers such that:

- $\cup\{U_\beta; \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha \subseteq cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$  for every  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;
- $\cup\{cl_Y V_\beta; \beta \in \pi_n^{-1}(\alpha)\} \subseteq V_\alpha$  and  $diam(V_\alpha) < 2^{-n}$  for every  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;

- $|A_n| < \tau_n \leq \tau_{n+1} < \tau$  for every  $n \in \mathbb{N}$ ;
- if  $A'_n = \{\alpha \in A_n : F \cap cl_X U_\alpha = \emptyset\}$  and  $A''_n = A_n \setminus A'_n$ , then  $|A''_n| < \tau^*$  and  $F \subseteq U_n \subseteq cl_X U_n \subseteq \cup\{U_\alpha : \alpha \in A''_n\}$ ;
- the family  $\gamma''_n = \{U_\alpha : \alpha \in A''_n\}$  is locally finite in  $X$  for every  $n \in \mathbb{N}$ ;
- $cl_X U_n \cap cl_X U_\alpha = \emptyset$  for every  $n \in \mathbb{N}$  and  $\alpha \in A'_n$ .

Let  $\eta = \{V : V \text{ is open in } Y \text{ and } diam(V) < 2^{-1}\}$  and  $\gamma' = \{U : U \text{ is open in } X \text{ and } cl_X U \subseteq \theta^{-1}(V) \text{ for some } V \in \eta\}$ . There exist a locally finite subfamily  $\gamma'_1 = \{U_\alpha : \alpha \in A'_1\}$  of  $\gamma'$  such that  $|A'_1| < \tau^* < k(F)$  and an open subset  $U_1$  of the space  $X$  such that  $F \subseteq U_1 \subseteq cl_X U_1 \subseteq \cup\{U_\alpha : \alpha \in A'_1\}$  and  $F \cap U_\alpha \neq \emptyset$  for every  $\alpha \in A'_1$ . For every  $\alpha \in A'_1$  fix  $V_\alpha \in \eta$  such that  $cl U_\alpha \subseteq \theta^{-1}(V_\alpha)$ . Let  $Y_1 = X \setminus U_1$  and  $\tau_1 = k(F) + \tau^*$ . Since  $k(Y) \leq \tau_1 < \tau$ , there exists an open subfamily  $\gamma'_1 = \{U_\alpha : \alpha \in A'_1\}$  of  $\gamma'$  such that  $|A'_1| \leq \tau_1$ ,  $Y_1 \subseteq \cup\{U_\alpha : \alpha \in A'_1\}$  and  $cl_X U_1 \cap (\cup\{cl_X U_\alpha : \alpha \in A'_1\}) = \emptyset$ . For every  $\alpha \in A'_1$  fix  $V_\alpha \in \eta'$  such that  $cl_X U_\alpha \subseteq \theta^{-1}(V_\alpha)$ . Let  $A_1 = A'_1 \cup A''_1$ ,  $\gamma_1 = \{U_\alpha : \alpha \in A_1\}$  and  $\eta_1 = \{V_\alpha : \alpha \in A_1\}$ .

The objects  $\{\gamma_1, \xi_1, U, \tau_1\}$  are constructed.

Consider that the objects  $\{\gamma_i, \xi_i, \pi_{i-1}, U_i, \tau_i, : i \leq n\}$  are constructed.

Fix  $\alpha \in A_n$ .

Let  $\eta_\alpha = \{V : V \text{ is open in } Y, cl_Y V \subseteq V_\alpha \text{ and } diam(V) < 2^{-n-1}\}$  and  $\gamma_\alpha^* = \{W : W \text{ is open in } X \text{ and } cl_X W \subseteq \theta^{-1}(V) \text{ for some } V \in \eta_\alpha\}$ .

Assume that  $\alpha \in A''_n$ .

Since  $F_\alpha = F \cap cl_X U_\alpha$  is a closed subset of  $X$ , then there exists a locally finite subfamily  $\gamma''_\alpha = \{W_\beta : \beta \in A''_\alpha\}$  of  $\gamma_\alpha^*$ , where  $|A''_\alpha| < \tau^*$  such that  $F_\alpha \subseteq \cup\{W_\beta : \beta \in A''_\alpha\}$ ,  $F_\alpha \cap W_\beta \neq \emptyset$  for every  $\beta \in A''_\alpha$  and for every  $\beta \in A''_\alpha$  there exists  $V_\beta \in \eta_\alpha$  such that  $cl_X W_\beta \subseteq \theta^{-1}(V_\beta)$ . We put  $U_\beta = U_\alpha \cap W_\beta$  for every  $\beta \in A''_\alpha$ .

Let  $A''_{n+1} = \cup\{A''_\alpha : \alpha \in A''_n\}$ ,  $\gamma''_{n+1} = \{U_\alpha : \alpha \in A''_{n+1}\}$  and  $\eta''_{n+1} = \{V_\alpha : \alpha \in A''_{n+1}\}$ .

The family  $\gamma''_{n+1}$  is locally finite.

Let  $\Phi_\alpha = cl_X U_\alpha \setminus \cup\{U_\beta : \beta \in A''_\alpha\}$  and  $U'_n = U_n \setminus \cup\{\Phi_\alpha : \alpha \in A''_n\}$ . Since the family  $\gamma''_n$  is locally finite, the set  $U'_n$  is open in  $X$  and  $F \subseteq U'_n \subseteq \cup\{U_\beta : \beta \in A''_\alpha\}$ .

There exists an open subset  $U_{n+1}$  of  $X$  such that  $U_{n+1} \subseteq cl_X U_{n+1} \subseteq \cup\{U_\beta : \beta \in A''_\alpha\}$ .

Let  $Y_{n+1} = X \setminus U_n$  and  $\tau_{n+1} = k(Y_{n+1}) + \tau_n$ .

For every  $\alpha \in A_n$  there exist the subfamily  $\gamma'_\alpha = \{W_\beta : \beta \in A'_\alpha\}$  of  $\gamma_\alpha^*$  and the subfamily  $\eta'_{i\alpha} = \{V_\beta : \beta \in A'_\alpha\}$  of  $\gamma'_\alpha$  such that:

- $|A'_\alpha| < \tau_{n+1}$ ;
- $cl_X U_\alpha \setminus U_n \subseteq \cup\{W_\beta : \beta \in A'_\alpha\}$ ;
- $cl_X W_\beta \cap cl_X U_{n+1} = \emptyset$  for any  $\beta \in A'_\alpha$ .

Now we put  $A_\alpha = A'_\alpha \cup A''_\alpha$ ,  $A_{n+1} = \cup\{A_\alpha : \alpha \in A_n\}$ ,  $U_\beta = U_\alpha \cap U_\beta$  for any  $\beta \in A_\alpha$ ,  $\gamma_{n+1} = \{U_\alpha : \alpha \in A_{n+1}\}$ ,  $\eta_{n+1} = \{V_\alpha : \alpha \in A_{n+1}\}$  and  $\pi_{n+1}^{-1}(\alpha) = A_{n\alpha}$ .

The objects  $\{\gamma_n, \xi_n, \pi_n, U_n, \tau_n : n \in \mathbb{N}\}$  are constructed.

Since  $\tau$  is not sequential, we have  $m = \sup\{\tau_n : n \in \mathbb{N}\} < \tau$ .

Let  $x \in X$ . Denote by  $A(x)$  the set of all sequences  $\alpha = (\alpha_n : n \in \mathbb{N})$  for which  $\alpha_n \in A_n$  and  $x \in U_{\alpha_n}$  for every  $n \in \mathbb{N}$ . For every  $\alpha = (\alpha_n : n \in \mathbb{N}) \in A(x)$  there exists a unique point  $y(\alpha) \in Y$  such that  $\{y(\alpha)\} = \cap\{V_{\alpha_n} : n \in \mathbb{N}\}$ . It is obvious that  $y(\alpha) \in \theta(x)$ . Let  $\phi(x) = \{y(\alpha) : \alpha \in A(x)\}$ . Then  $\phi$  is a selection of  $\theta$ . By construction:

- $U_\alpha \subseteq \phi^{-1}(V_\alpha)$  for all  $\alpha \in A_n$  and  $n \in \mathbb{N}$ ;
- the mapping  $\phi$  is lower semi-continuous;
- if  $Z = \phi(X)$ , then  $\{H_\alpha = Z \cap V_\alpha : \alpha \in A = \cup\{A_n : n \in \mathbb{N}\}\}$  is an open base of the subspace  $Z$  and  $w(Z) \leq m$ .

Thus we have proved the implication  $(K1 \rightarrow K11)$ .

Lemma 2.1 completes the proof of the theorem.  $\square$

**Remark 2.8.** Let  $X$  be a paracompact space and  $Y \subseteq X$ . Then  $l(cl_X Y) \leq l(Y)$  and  $k(cl_X Y) \leq k(Y)$ .

Theorem 2.7, Corollary 2.6 and Lemma 1.4 yield

**Corollary 2.9.** Let  $X$  be a paracompact and  $\tau$  be an infinite cardinal. Then the properties  $K1 - K10$  are equivalent.

One can observe that the Corollary 2.9 follows from Proposition 2.2, Lemma 2.1 and Theorem 0.1, too.

**Corollary 2.10.** Let  $X$  be a space and  $\tau$  be an uncountable not sequential cardinal number. Then the following assertions are equivalent:

1.  $X$  is a paracompact space and  $k(X) \leq \tau$ .
2.  $X$  is a paracompact space and for every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $w(\phi(X)) < \tau$ .
3.  $X$  is a paracompact space and for every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  such that  $w(g(X)) < \tau$ .
4.  $X$  is a paracompact space and for every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  such that  $|g(X)| < \tau$ .
5. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exist a compact-valued lower semi-continuous mapping  $\varphi : X \rightarrow Y$  and a compact-valued upper semi-continuous mapping  $\psi : X \rightarrow Y$  such that  $w(\psi(X)) < \tau$  and  $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$  for any  $x \in X$ .
6. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists an upper semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $w(\phi(X)) < \tau$ .

**Example 2.11.** Let  $\tau$  be an uncountable limit cardinal number and  $m = cf(\tau)$ . Fix a well ordered set  $A$  and a family of regular cardinal numbers  $\{\tau_\alpha : \alpha \in A\}$  such that  $\sup\{\tau_\alpha : \alpha \in A\} = \tau$  and  $\tau_\alpha < \tau_\beta < \tau$  for all  $\alpha, \beta \in A$  and  $\alpha < \beta$ . For every  $\alpha \in A$  fix a zero-dimensional complete metric space  $X_\alpha$  such

that  $w(X_\alpha) = \tau_\alpha$ . Let  $X'$  be the discrete sum of the spaces  $\{X_\alpha : \alpha \in A\}$ . Then  $X'$  is a complete metrizable space and  $w(X') = \tau$ . Thus  $l(X') = \tau$  and  $k(X') = \tau^+$ . Fix a point  $b \notin X'$ . Put  $X = \{b\} \cup X'$  with the topology generated by the open bases  $\{U \subseteq X' : U \text{ is open in } X'\} \cup \{X \setminus \bigcup\{X_\beta : \beta \leq \alpha\} : \alpha \in A\}$ . Then  $X$  is a zero-dimensional paracompact space and  $\chi(X) = \chi(b, X) = cf(\tau)$ . If  $cf(\tau) = \aleph_0$ , then  $X$  is a complete metrizable space. If  $Y \subseteq X'$  is a closed subspace of  $X$ , then there exists  $\alpha \in A$  such that  $Y \subseteq \bigcup\{X_\beta : \beta < \alpha\}$ ,  $w(Y) < \tau_\alpha$  and  $k(Y) < \tau$ . Therefore  $k(X) = \tau$ .

Let  $Z = X \times [0, 1]$ . Then  $k(Z) = \tau$  and  $k(Z, \tau) = \{b\} \times [0, 1]$ .

Suppose that  $\tau$  is not a sequential cardinal number,  $\mathbb{N}$  is a discrete space and  $S = X \times \mathbb{N}$ . Then  $k(S) = \tau$  and  $k(S, \tau) = \{b\} \times \mathbb{N}$ .

Moreover, if  $m = cf(\tau)$  is uncountable,  $X_\tau$  is a complete metrizable space,  $w(X_\tau) < m$  and  $Z_\tau = X \times X_\tau$ , then  $k(Z_\tau) = \tau$  and  $k(Z_\tau, \tau) = \{b\} \times X_\tau$ .  $\square$

### 3 On the geometry of paracompact spaces

Let  $\Pi$  be the class of all paracompact spaces.

For every infinite cardinal number  $\tau$  we denote by  $\Pi(\tau)$  the class  $\{X \in \Pi : k(cw(X)) \leq \tau\}$ . We put  $\Pi_l(\tau) = \{X \in \Pi : l(cw(X)) \leq \tau\}$ .

It is obvious that  $\Pi(\tau) \subseteq \Pi_l(\tau) \subseteq \Pi(\tau^+)$ .

We consider that  $\Pi(n) = \{X \in \Pi : \dim X = 0\}$  for any  $n \in \{0\} \cup \mathbb{N}$ .

Our aim is to prove that the classes  $\Pi(\tau)$  may be characterized in terms of selections. The main results of the section are the following two theorems.

**Theorem 3.1.** *Let  $X$  be a space and  $\tau$  be an uncountable non-sequential cardinal number. Then the following assertions are equivalent:*

1.  $X \in \Pi(\tau)$ , i.e.  $X$  is paracompact and  $k(cw(X)) \leq \tau$ .
2.  $X$  is a paracompact space and for every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $w(\phi(cw(X))) < \tau$ .
3.  $X$  is a paracompact space and for every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  such that  $w(g(cw(X))) < \tau$ .
4.  $X$  is a paracompact space and for every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  such that  $|g(cw(X))| < \tau$ .
5. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exist a compact-valued lower semi-continuous mapping  $\varphi : X \rightarrow Y$  and a compact-valued upper semi-continuous mapping  $\psi : X \rightarrow Y$  such that  $w(\psi(X)) < \tau$  and  $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$  for any  $x \in cw(X)$ .
6. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exist a closed  $G_\delta$ -set  $H$  of  $X$  and an upper semi-continuous compact-valued selection  $\psi : X \rightarrow Y$  such that:
  - i)  $cw(X) \subseteq H$  and  $w(\psi(H)) < \tau$ ;
  - ii)  $\psi(x)$  is a one-point set of  $Y$  for every  $x \in X \setminus H$ ;
  - iii)  $cl_Y \psi(H) = cl_Y \psi(cw(X))$ .



7. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exists an upper semi-continuous compact-valued selection  $\psi : X \rightarrow Y$  such that  $k(\psi^\infty(x)) < \tau$  for every  $x \in X$ .

8. For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists an upper semi-continuous selection  $\psi : X \rightarrow Y$  such that  $|\psi^\infty(x)| < \tau$  for every  $x \in X$ .

**Theorem 3.2.** Let  $X$  be a space and  $\tau$  be an infinite cardinal number. Then the following assertions are equivalent:

1.  $X \in \Pi(\tau)$ , i.e.  $X$  is paracompact and  $k(c\omega(X)) \leq \tau$ .
2.  $X$  is a paracompact space and for every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a lower semi-continuous selection  $\phi : X \rightarrow Y$  of  $\theta$  such that  $k(cl_Y \phi(c\omega(X))) \leq \tau$ .
3.  $X$  is a paracompact space and for every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  such that  $k(cl_Y g(c\omega(X))) \leq \tau$ .
4.  $X$  is a paracompact space and for every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists a single-valued selection  $g : X \rightarrow Y$  such that  $|g(c\omega(X))| < \tau$ .
5. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metrizable space  $Y$  there exist a compact-valued lower semi-continuous mapping  $\varphi : X \rightarrow Y$  and a compact-valued upper semi-continuous mapping  $\psi : X \rightarrow Y$  such that  $k(cl_Y(\psi(c\omega(X)))) \leq k(\psi(c\omega(X))) \leq \tau$  and  $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$  for any  $x \in c\omega(X)$ .
6. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exist a closed  $G_\delta$ -set  $H$  of  $X$  and an upper semi-continuous compact-valued selection  $\psi : X \rightarrow Y$  such that:
  - i)  $c\omega(X) \subseteq H$  and  $k(\psi(H)) \leq \tau$ ;
  - ii)  $\psi(x)$  is a one-point set of  $Y$  for every  $x \in X \setminus H$ ;
  - iii)  $cl_Y \psi(H) = cl_Y \psi(c\omega(X))$ .
7. For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exists an upper semi-continuous compact-valued selection  $\psi : X \rightarrow Y$  such that  $k(\psi^n(x)) < \tau$  for every  $x \in X$  and any  $n \in \mathbb{N}$ .
8. For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists an upper semi-continuous selection  $\psi : X \rightarrow Y$  such that  $|\psi^n(x)| < \tau$  for every  $x \in X$  and any  $n \in \mathbb{N}$ .

**Proof of the Theorems:** Let  $X \in \Pi(\tau)$  and  $\theta : X \rightarrow Y$  be a lower semi-continuous closed-valued mapping into a complete metric space  $(Y, d)$ . For every subset  $L$  of  $Y$  and every  $n \in \mathbb{N}$  we put  $O(L, n) = \{y \in Y : d(y, L) = \inf\{d(x, z) : z \in L\} < 2^{-n}\}$ . Obviously,  $cl_Y L = \bigcap \{O(L, n) : n \in \mathbb{N}\}$  and  $cl_Y O(L, n+1) \subseteq O(L, n)$  for any  $n \in \mathbb{N}$ .

By virtue of the Michael's Theorem 0.1, there exist a compact-valued lower semi-continuous mapping  $\varphi : X \rightarrow Y$  and a compact-valued upper semi-continuous mapping  $\psi : X \rightarrow Y$  such that  $\varphi(x) \subseteq \psi(x) \subseteq \theta(x)$  for any  $x \in c\omega(X)$ .

From Proposition 2.2 it follows that  $k(cl_Y(\psi(c\omega(X)))) \leq k(\psi(c\omega(X))) \leq \tau$  and  $k(cl_Y(\varphi(c\omega(X)))) \leq k(cl_Y(\psi(c\omega(X)))) \leq \tau$ . Moreover, if  $\tau$  is a not sequential cardinal number, then  $w(\varphi(c\omega(X))) \leq w(\psi(c\omega(X))) < \tau$ .

Therefore, the assertions 2, 3, 4 and 5 of Theorems follow from the assertion 1.

It will be affirmed that there exist a sequence  $\{\phi_n : X \rightarrow Y : n \in \mathbb{N}\}$  of lower semi-continuous compact-valued mappings, a sequence  $\{\psi_n : X \rightarrow Y : n \in \mathbb{N}\}$  of upper semi-continuous compact-valued mappings, a sequence  $\{V_n : n \in \mathbb{N}\}$  of open subsets of  $Y$  and a sequence  $\{H_n : n \in \mathbb{N}\}$  of open-and-closed subsets of  $X$  such that:

- 1)  $\psi_{n+1}(x) \subseteq \phi_n(x) \subseteq \psi_n(x) \subseteq \theta(x)$  for every  $x \in X$  and every  $n \in \mathbb{N}$ ;
- 2)  $\phi_n(x) = \psi_n(x)$  is a one-point subset of  $Y$  for every  $x \in X \setminus H_n$  and for every  $n \in \mathbb{N}$ ;
- 3)  $H_{n+1} \subseteq \{x \in X : \psi_n(x) \subseteq V_n, H_{n+1} \subseteq H_n \text{ and } V_{n+1} = O(\psi_n(c\omega(X)))$  for every  $n \in \mathbb{N}$ ;

Let  $V_1 = O(\theta(c\omega(X)))$  and  $U_1 = \theta^{-1}V_1$ . From Lemma 0.2 it follows that there exists an open-and-closed subset  $H_1$  of  $X$  such that  $c\omega(X) \subseteq H_1 \subseteq U_1$ .

Since  $\dim(X \setminus H_1) = 0$  there exists a single-valued continuous mapping  $h_1 : X \setminus H_1 \rightarrow Y$  such that  $h_1(x) \in \theta(x)$  for every  $x \in X \setminus H_1$ . Since  $H_1$  is a paracompact space,  $V_1$  is a complete metrizable space and  $\theta_1 : H_1 \rightarrow V_1$ , where  $\theta_1(x) = V_1 \cap \theta(x)$ , is a lower semicontinuous closed-valued in  $V_1$  mapping, by virtue of Theorem 0.1, there exist a compact-valued lower semi-continuous mapping  $\varphi_1 : H_1 \rightarrow V_1$  and a compact-valued upper semi-continuous mapping  $\lambda_1 : H_1 \rightarrow V_1$  such that  $\varphi_1(x) \subseteq \lambda_1(x) \subseteq \theta_1(x)$  for any  $x \in H_1$ .

Put  $\psi_1(x) = \phi_1(x) = h_1(x)$  for  $x \in X \setminus H_1$  and  $\psi_1(x) = \lambda_1(x)$ ,  $\phi_1(x) = \varphi_1(x)$  for  $x \in H_1$ .

The objects  $\phi_1$  and  $\psi_1$  are constructed.

Suppose that  $n > 1$  and the objects  $\phi_{n-1}, \psi_{n-1}, H_{n-1}$  and  $V_{n-1}$  had been constructed.

We put  $F_n = cl_Y \psi_{n-1}(c\omega(X))$ ,  $V_n = O(F_n, n)$  and  $U_n = \{x \in H_{n-1} : \psi_{n-1}(x) \subseteq V_n\}$ . From Lemma 0.2 it follows that there exists an open-and-closed subset  $H_n$  of  $X$  such that  $c\omega(X) \subseteq H_n \subseteq U_n$ .

Since  $\dim(X \setminus H_n) = 0$  there exists a single-valued continuous mapping  $h_n : X \setminus H_n \rightarrow Y$  such that  $h_n(x) \in \phi_{n-1}(x)$  for every  $x \in X \setminus H_n$ . By construction, we have  $\phi_{n-1} \subseteq \psi(x) \subseteq V_n$  for any  $x \in H_n$ . Since  $H_n$  is a paracompact space,  $V_n$  is a complete metrizable space and  $\theta_n : H_n \rightarrow V_n$ , where  $\theta_n(x) = V_n \cap \phi_{n-1}(x)$ , is a lower semicontinuous closed-valued in  $V_n$  mapping, by virtue of Theorem 0.1, there exist a compact-valued lower semi-continuous mapping  $\varphi_n : H_n \rightarrow V_n$  and a compact-valued upper semi-continuous mapping  $\lambda_n : H_n \rightarrow V_n$  such that  $\varphi_n(x) \subseteq \lambda_n(x) \subseteq \theta_n(x)$  for any  $x \in H_n$ .

Put  $\psi_n(x) = \phi_n(x) = h_n(x)$  for  $x \in X \setminus H_n$  and  $\psi_n(x) = \lambda_n(x)$ ,  $\phi_n(x) = \varphi_n(x)$  for  $x \in H_n$ . The objects  $\phi_n$  and  $\psi_n$  are constructed.

Now we put  $\lambda(x) = \cap\{\psi_n(x) : n \in \mathbb{N}\}$  for any  $x \in X$  and  $H = \cap\{H_n : n \in \mathbb{N}\}$ .

Since  $\lambda^{-1}(\Phi) = \cap\{\psi_n^{-1}(\Phi) : n \in \mathbb{N}\}$  for any closed subset  $\Phi$  of  $Y$ , the mapping  $\lambda$  is compact-valued and upper semi-continuous. By construction,

- i)  $c\omega(X) \subseteq H$  and  $k(\lambda(H)) \leq \tau$ ;
- ii)  $\lambda(x)$  is a one-point set of  $Y$  for every  $x \in X \setminus H$ ;
- iii)  $cl_Y \lambda(H) = cl_Y \lambda(c\omega(X))$ ;

iv)  $\lambda(\lambda^{-1}(A)) \subseteq A \cup \lambda(H)$  for every subset  $A$  of  $Y$ .

Therefore, the assertions 6, 7 and 8 of Theorems follow from the assertion 1.

(8  $\Rightarrow$  1) Let  $\gamma = \{U_\alpha : \alpha \in A\}$  be an open cover of  $X$ . On  $A$  introduce the discrete topology and put  $\theta(x) = \{\alpha \in A : x \in U_\alpha\}$  for  $x \in X$ . Since  $\theta^{-1}(H) = \bigcup \{U_\alpha : \alpha \in H\}$  for every subset  $H$  of  $A$ , the mapping  $\theta : X \rightarrow A$  is lower semi-continuous. Let  $\psi : X \rightarrow A$  be an upper semi-continuous selection of  $\theta$  with  $|\psi^2(x)| < \tau$  for every  $x \in X$ . Then  $\xi = \{\Psi_\alpha = \psi^{-1}(\alpha) : \alpha \in A\}$  is a closed closure-preserving  $\tau^-$ -star shrinking of the cover  $\xi$ . By virtue of Proposition 1.5, the assertion 1 follows from the assertion 8.  $\square$

**Corollary 3.3.** *For a topological space  $X$  the following assertions are equivalent:*

- 1)  $X$  is paracompact and  $c\omega(X)$  is compact.
- 2)  $X$  is strongly paracompact and  $c\omega(X)$  is compact.
- 3) For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exist an upper semi-continuous compact-valued selection  $\psi : X \rightarrow Y$  and a closed  $G_\delta$ -subset  $H$  of  $X$  such that  $c\omega(X) \subseteq H$ ,  $cl_Y(\psi(H))$  is compact and  $\psi(x)$  is a one-point set for every  $x \in X \setminus H$ .
- 4) For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exists an upper semi-continuous selection  $\psi : X \rightarrow Y$  such that  $cl_Y \psi^\infty(x)$  is compact for every  $x \in X$ .
- 5) For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists an upper semi-continuous selection  $\psi : X \rightarrow Y$  such that the set  $\psi^\infty(x)$  is finite for every  $x \in X$ .
- 6) For every open cover of  $X$  there exists an open star-finite shrinking.

**Proof:** For the implication (1  $\Rightarrow$  2) see Proposition 4, [4].

For the implications (1  $\Leftrightarrow$  6) see Proposition 5, [4].  $\square$

**Corollary 3.4.** *For a space and an infinite cardinal number  $\tau$  the following assertions are equivalent:*

- 1)  $X$  is paracompact and  $l(c\omega(X)) \leq \tau$ .
- 2) For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exist an upper semi-continuous compact-valued selection  $\psi : X \rightarrow Y$  and a closed  $G_\delta$ -subset  $H$  of  $X$  such that  $c\omega(X) \subseteq H$  and  $w(\psi(H)) \leq \tau$ ;  $\psi(x)$  is a one-point set for every  $x \in X \setminus H$ .
- 3) For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exists an upper semi-continuous compact-valued selection  $\psi : X \rightarrow Y$  such that  $w(\psi^\infty(x)) \leq \tau$  for every  $x \in X$ .
- 4) For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists an upper semi-continuous selection  $\psi : X \rightarrow Y$  such that  $|\psi^\infty(x)| \leq \tau$  for every  $x \in X$ .

**Corollary 3.5.** *For a topological space  $X$  the following assertions are equivalent:*

- 1)  $X$  is paracompact and  $c\omega(X)$  is Lindelöf.
- 2)  $X$  is strongly paracompact and  $c\omega(X)$  is Lindelöf.
- 3) For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exist an upper semi-continuous compact-valued

selection  $\psi : X \rightarrow Y$  and a closed  $G_\delta$ -subset  $H$  of  $X$  such that  $c\omega(X) \subseteq H$ ,  $\psi(H)$  is separable and  $\psi(x)$  is a one-point set for every  $x \in X \setminus H$ .

4) For every lower semi-continuous closed-valued mapping  $\theta : X \rightarrow Y$  into a complete metric space  $Y$  there exists an upper semi-continuous compact-valued selection  $\psi : X \rightarrow Y$  such that  $\psi^\infty(x)$  is separable for every  $x \in X$ .

5) For every lower semi-continuous mapping  $\theta : X \rightarrow Y$  into a discrete space  $Y$  there exists an upper semi-continuous selection  $\psi : X \rightarrow Y$  such that the set  $\psi^\infty(x)$  is countable for every  $x \in X$ .

6) For every open cover of  $X$  there exists an open star-countable shrinking.

**Example 3.6.** Let  $A$  be an uncountable set and  $X_\alpha$  be a non-empty compact space for every  $\alpha \in A$ . Let  $X = \bigoplus \{X_\alpha : \alpha \in A\}$  be the discrete sum of the space  $\{X_\alpha : \alpha \in A\}$ . Let  $B = \{\alpha \in A : \dim X_\alpha \neq 0\}$ . Then  $c\omega(X)$  is compact if and only if the set  $B$  is finite. If the set  $B$  is infinite then  $l(c\omega(X)) = |B|$  and  $k(c\omega(X)) = |B|^+$ .  $\square$

**Example 3.7.** Let  $\tau$  be an uncountable non-sequential cardinal number. Fix an infinite set  $A_m$  for every cardinal number  $m < \tau$  assuming that  $A_m \cap A_n = \emptyset$  for  $m \neq n$ . Put  $A = \bigcup \{A_m : m < \tau\}$ . Let  $\{X_\alpha : \alpha \in A\}$  be a family of non-empty compact spaces assuming that  $X_\alpha \cap X_\beta = \emptyset$  for  $\alpha \neq \beta$ . Put  $B_m = \{\alpha \in A_m : \dim X_\alpha \neq 0\}$  and  $1 \leq |B_m| \leq m$  for every  $m < \tau$ . Fix a point  $b \notin \bigcup \{X_\alpha : \alpha \in A\}$ . Let  $X = \{b\} \cup (\bigcup \{X_\alpha : \alpha \in A\})$ . Suppose that  $X_\alpha$  is an open subset of  $X$  and  $\{H_m = \{b\} \cup (\bigcup \{X_\alpha : \alpha \in A_n, n \leq m\}) : m < \tau\}$  is a base of  $X$  at  $b$ . If  $Z = \{b\} \cup (\bigcup \{X_\alpha : \alpha \in B_m, m < \tau\})$ , then  $c\omega(X) \subseteq Z$  and  $k(c\omega(X)) \leq k(Z) = l(Z) = \tau$ .  $\square$

**Example 3.8.** Let  $\tau$  be a regular uncountable cardinal number,  $A$  be an infinite set,  $\tau < |A|$ ,  $\{X_\alpha : \alpha \in A\}$  be a family of non-empty compact spaces,  $X_\alpha \cap X_\beta = \emptyset$  for  $\alpha \neq \beta$ ,  $B = \{\alpha \in A : \dim X_\alpha \neq 0\}$ ,  $\tau = |B|$  and  $b \notin \bigcup \{X_\alpha : \alpha \in A\}$ . Let  $X = \{b\} \cup (\bigcup \{X_\alpha : \alpha \in A\})$ . Suppose that  $X_\alpha$  is an open subset of  $X$  and  $\{U_H = X \setminus \bigcup \{X_\alpha : \alpha \in H\} : H \subseteq A, |H| < \tau\}$  is a base of  $X$  at  $b$ . If  $Z = \{b\} \cup (\bigcup \{X_\alpha : \alpha \in B\})$ , then  $c\omega(X) \subseteq Z$  and  $k(c\omega(X)) \leq k(Z) = l(Z) = \tau$ .  $\square$

**Example 3.9.** Let  $\tau$  be a regular uncountable limit cardinal number and  $2^m < \tau$  for any  $m < \tau$ . Let  $\{m_\alpha : \alpha \in A\}$  be a family of infinite cardinal numbers such that  $|A| = \tau$ , the set  $A$  is well ordered and  $m_\alpha < m_\beta$ ,  $|\{\mu \in A : \mu \leq \alpha\}| < \tau$  provided  $\alpha, \beta \in A$  and  $\alpha < \beta$ . For any  $\alpha \in A$  fix a discrete space of the cardinality  $m_\alpha$ . Let  $X = \prod \{X_\alpha : \alpha \in A\}$ . If  $x = (x_\alpha : \alpha \in A) \in X$  and  $\beta \in A$ , then  $O(\beta, x) = \{y = (y_\alpha : \alpha \in A) \in X : y_\alpha = x_\alpha \text{ for any } \alpha \leq \beta\}$ . The family  $\{O(\beta, x) : \beta \in A, x \in X\}$  form the open base of the space  $X$ . The space  $X$  is paracompact and  $w(X) = l(X) = \tau$ . It is obvious that  $c(X, \tau) = X$ , we have  $k(X) = \tau^+$ . If  $\alpha \in A$ , then  $\gamma_\alpha = \{O(\alpha, x) : x \in X\}$  is open discrete cover of  $X$  and  $|\gamma_\alpha| = 2^{m_\alpha} < \tau$ .  $\square$

## 4 On the class $\Pi(0)$ of spaces

In the present section the class of all paracompact spaces  $X$  such that  $\dim X = 0$  is studied.

**Definition 4.1** A set-valued mapping  $\psi : X \longrightarrow Y$  is called virtual single-valued if  $\psi^\infty(x) = \psi(x)$  for every  $x \in X$ .

**Remark 4.2** It is obvious that for a set-valued mapping  $\theta : X \longrightarrow Y$  the following conditions are equivalent:

1.  $\psi$  is a virtual single-valued mapping;
2.  $\psi^2(x) = \psi(x)$  for every  $x \in X$  ;
3.  $\psi^n(x) = \psi(x)$  for every  $x \in X$  and some  $n \geq 2$ ;
4.  $\psi(x) = \psi(y)$  provided  $x, y \in X$  and  $\psi(x) \cap \psi(y) \neq \emptyset$ .
5.  $\psi^{-1}(y) = \psi^{-1}(z)$  provided  $y, z \in Y$  and  $\psi^{-1}(y) \cap \psi^{-1}(z) \neq \emptyset$ .

Note that, if  $f : X \longrightarrow Y$  is a single-valued mapping onto a space  $Y$ , then  $f^{-1}$  and  $f$  are virtual single-valued mappings.

Denote with  $D = \{0, 1\}$  the two-point discrete space.

**Theorem 4.3** For a space  $X$ , the following assertions are equivalent:

1.  $X$  is normal and  $\dim X = 0$ ;
2. For every lower semi-continuous mapping  $\theta : X \longrightarrow D$  there exists a virtual single-valued lower semi-continuous selection;
3. For every lower semi-continuous mapping  $\theta : X \longrightarrow D$  there exists a virtual single-valued upper semi-continuous selection;
4. For every lower semi-continuous mapping  $\theta : X \longrightarrow D$  there exists a single-valued continuous selection.

**Proof:** Implications  $(1 \Leftrightarrow 4)$  is a well known fact. Implications  $(4 \Rightarrow 2)$  and  $(4 \Rightarrow 3)$  are obvious as every single-valued continuous selection is virtual single-valued.

$(2 \Rightarrow 1)$  and  $(3 \Rightarrow 1)$  Let  $F_1$  and  $F_2$  be two disjoint closed subsets of  $X$ . Put  $\theta(x) = \{0\}$  for  $x \in F_1$ ,  $\theta(x) = \{1\}$  for  $x \in F_2$  and  $\theta(x) = \{0, 1\}$  for  $x \in X \setminus (F_1 \cup F_2)$ . The mapping  $\theta : X \longrightarrow D$  is lower semi-continuous. Suppose that  $\lambda : X \longrightarrow D$  is a virtual single-valued selection of  $\theta$ . Put  $H_1 = \lambda^{-1}(0)$  and  $H_2 = \lambda^{-1}(1)$ . Then  $F_1 \subseteq H_1$  and  $F_2 \subseteq H_2$ ,  $X = H_1 \cup H_2$  and  $H_1 \cap H_2 = \emptyset$ . If  $\lambda$  is lower semi-continuous (or upper semi-continuous) the sets  $H_1, H_2$  are open (closed).  $\square$

Let  $\tau$  be an infinite cardinal number. A topological space  $X$  is called  $\tau$ -paracompact if  $X$  is normal and every open cover of  $X$  of the cardinality  $\leq \tau$  has a locally finite open refinement.

**Theorem 4.4** For a space  $X$  and an infinite cardinal number  $\tau$  the following assertions are equivalent:

1.  $X$  is a  $\tau$ -paracompact space and  $\dim X = 0$ .
2. For every lower semi-continuous mapping  $\theta : X \longrightarrow Y$  into a complete metrizable space  $Y$  of the weight  $\leq \tau$  there exists a virtual single-valued lower semi-continuous selection;
3. For every lower semi-continuous mapping  $\theta : X \longrightarrow Y$  into a complete metrizable space  $Y$  of the weight  $\leq \tau$  there exists a virtual single-valued upper semi-continuous selection;
4. For every lower semi-continuous mapping  $\theta : X \longrightarrow Y$  into a complete metrizable space  $Y$  of the weight  $\leq \tau$  there exists a single-valued continuous selection;

5. For every lower semi-continuous mapping  $\theta : X \longrightarrow Y$  into a discrete space  $Y$  of the cardinality  $\leq \tau$  there exists a single-valued continuous selection.

**Proof:** Let  $\gamma = \{U_\alpha : \alpha \in A\}$  be an open cover of  $X$  and  $|A| \leq \tau$ . Consider that  $A$  is a wellordered discrete space and  $\theta(x) = \{\alpha \in A : x \in U_\alpha\}$  for any  $x \in X$ . Then  $\theta$  is a lower semi-continuous mapping. Suppose that  $\psi : X \rightarrow Y$  is a virtual single-valued lower or upper semi-continuous selection of  $\theta$ . For any  $x \in X$  we denote by  $f(x)$  the first element of the set  $\psi(x)$ . Then  $f : X \rightarrow Y$  is a single-valued continuous selection of the mappings  $\theta$  and  $\psi$ . Therefore  $H_\alpha = f^{-1}(\alpha) : \alpha \in A$  is a discrete refinement of  $\gamma$ . The implications  $(2 \Rightarrow 1)$ ,  $(2 \Rightarrow 4)$ ,  $(3 \Rightarrow 1)$ ,  $(3 \Rightarrow 4)$  and  $(5 \Rightarrow 1)$  are proved. The implications  $(4 \Rightarrow 5)$ ,  $(4 \Rightarrow 2)$  and  $(4 \Rightarrow 3)$  are obvious. The implication  $(1 \Rightarrow 4)$  is wellknown (see [1, 2]).  $\square$

**Corollary 4.5** For a space  $X$  the following assertions are equivalent:

1.  $X$  is a paracompact space and  $\dim X = 0$ .
2. For every lower semi-continuous mapping  $\theta : X \longrightarrow Y$  into a complete metrizable space  $Y$  there exists a virtual single-valued lower semi-continuous selection;
3. For every lower semi-continuous mapping  $\theta : X \longrightarrow Y$  into a complete metrizable space  $Y$  there exists a virtual single-valued upper semi-continuous selection;
4. For every lower semi-continuous mapping  $\theta : X \longrightarrow Y$  into a complete metrizable space  $Y$  there exists a single-valued continuous selection;
5. For every lower semi-continuous mapping  $\theta : X \longrightarrow Y$  into a discrete space  $Y$  there exists a single-valued continuous selection.

**Remark 4.6** Let  $Y$  be a topological space. Then:

1. If the space  $Y$  is discrete, then every lower semi-continuous virtual single-valued mapping or every upper semi-continuous virtual single-valued mapping  $\theta : X \longrightarrow Y$  into the space  $Y$  is continuous.
2. If the space  $Y$  is not discrete, then there exist a paracompact space  $X$  and a virtual single-valued mapping  $\theta : X \longrightarrow Y$  such that;
  - $\theta$  is upper semi-continuous and not continuous;
  - $X$  has a unique not isolated point.
3. If  $Y$  has an open non-discrete subspace  $U$  and  $|U| \leq |Y \setminus U|$ , then there exist a paracompact space  $X$  and a virtual single-valued mapping  $\theta : X \longrightarrow Y$  such that;
  - $\theta$  is lower semi-continuous and not continuous;
  - $X$  has a unique not isolated point.

**Remark 4.7** Let  $\gamma = \{Hy : y \in Y\}$  be a cover of a space  $X$ ,  $Y$  be a discrete space and  $\theta_\gamma(x) = \{y \in Y : x \in Hy\}$ . Then:

- the mapping  $\theta_\gamma$  is lower semi-continuous if and only if  $\gamma$  is an open cover;
- the mapping  $\theta_\gamma$  is upper semi-continuous if and only if  $\gamma$  is a closed and conservative cover;
- the mapping  $\varphi : X \rightarrow Y$  is a selection of the mapping  $\theta_\gamma$  if and only if  $\{Vy = \varphi^{-1}(y) : y \in Y\}$  is a shrinking of  $\gamma$ .

Therefore, the study of the problem of the selections for the mappings into discrete spaces is an essential case of the this problem.

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